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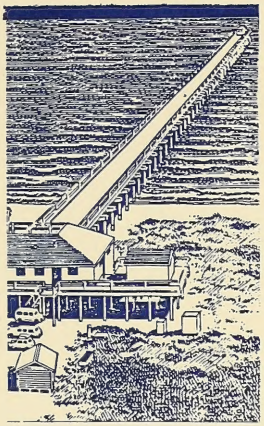

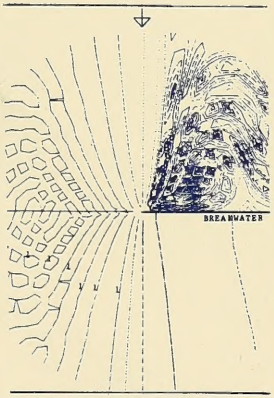


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TECHNICAL REPORT CERC-87-16

# COMBINED REFLECTION AND DIFFRACTION BY A VERTICAL WEDGE

by

H. S. Chen

Coastal Engineering Research Center

DEPARTMENT OF THE ARMY
Waterways Experiment Station, Corps of Engineers
PO Box 631, Vicksburg, Mississippi 39180-0631

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December 1987
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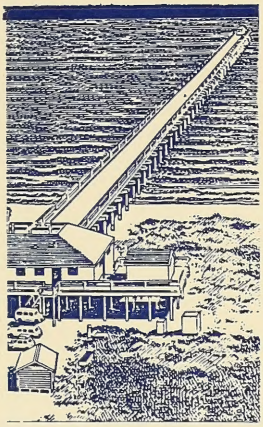

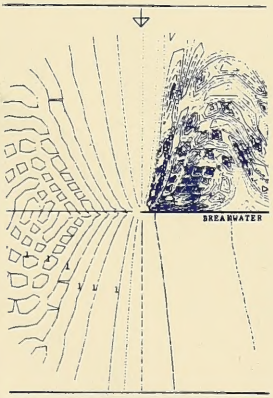


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SECURITY CLASSIFICATION OF THIS PAGE

## REPORT DOCUMENTATION PAGE

Form Approved
OMB No. 0704-0188
Exp. Date: Jun 30, 1986

| | |
|---|---|
| 1a. REPORT SECURITY CLASSIFICATION<br>Unclassified | 1b. RESTRICTIVE MARKINGS |
| 2a. SECURITY CLASSIFICATION AUTHORITY | 3. DISTRIBUTION / AVAILABILITY OF REPORT<br>Approved for public release; distribution unlimited. |
| 2b. DECLASSIFICATION / DOWNGRADING SCHEDULE | |
| 4. PERFORMING ORGANIZATION REPORT NUMBER(S)<br>Technical Report CERC-87-16 | 5. MONITORING ORGANIZATION REPORT NUMBER(S) |
| 6a. NAME OF PERFORMING ORGANIZATION<br>USAEWES, Coastal Engineering Research Center<br>6b. OFFICE SYMBOL (*If applicable*) | 7a. NAME OF MONITORING ORGANIZATION |
| 6c. ADDRESS (*City, State, and ZIP Code*)<br>PO Box 631<br>Vicksburg, MS 39180-0631 | 7b. ADDRESS (*City, State, and ZIP Code*) |
| 8a. NAME OF FUNDING / SPONSORING ORGANIZATION<br>US Army Corps of Engineers<br>8b. OFFICE SYMBOL (*If applicable*) | 9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER |
| 8c. ADDRESS (*City, State, and ZIP Code*)<br>Washington, DC 20314-1000 | 10. SOURCE OF FUNDING NUMBERS<br>PROGRAM ELEMENT NO. / PROJECT NO. / TASK NO. / WORK UNIT ACCESSION NO 31673 |

11. TITLE (*Include Security Classification*)

Combined Reflection and Diffraction by a Vertical Wedge

12. PERSONAL AUTHOR(S)
Chen, H. S.

| 13a. TYPE OF REPORT | 13b. TIME COVERED | 14. DATE OF REPORT (*Year, Month, Day*) | 15. PAGE COUNT |
|---|---|---|---|
| Final report | FROM _____ TO _____ | December 1987 | 75 |

16. SUPPLEMENTARY NOTATION
Available from National Technical Information Service, 5285 Port Royal Road, Springfield, VA 22161.

| 17. COSATI CODES | | | 18. SUBJECT TERMS (*Continue on reverse if necessary and identify by block number*) |
|---|---|---|---|
| FIELD | GROUP | SUB-GROUP | Breakwaters (LC)<br>Numerical analysis (LC)<br>Water waves (LC) |
| | | | |
| | | | |

19. ABSTRACT (*Continue on reverse if necessary and identify by block number*)

This report presents an analytical solution for the combined wave reflection and diffraction by a vertical wedge, of arbitrary wedge angle, subject to the excitation of a plane simple harmonic wave train coming from a far field. Results of two special cases are calculated: one a thin semi-infinite breakwater and the other a wedge of 90-deg angle. The results are presented in amplification factor diagrams. A subroutine WEDGE is also documented in the appendix. The subroutine can be used to calculate the case of arbitrary wedge angle.

| 20. DISTRIBUTION / AVAILABILITY OF ABSTRACT | 21. ABSTRACT SECURITY CLASSIFICATION |
|---|---|
| ☒ UNCLASSIFIED/UNLIMITED ☐ SAME AS RPT. ☐ DTIC USERS | Unclassified |
| 22a. NAME OF RESPONSIBLE INDIVIDUAL | 22b. TELEPHONE (*Include Area Code*) / 22c. OFFICE SYMBOL |

**DD FORM 1473,** 84 MAR — 83 APR edition may be used until exhausted. All other editions are obsolete.

SECURITY CLASSIFICATION OF THIS PAGE
Unclassified

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# PREFACE

The work in this report was authorized by the Office, Chief of Engineers (OCE), Coastal Engineering Functional Area of Civil Works Research and Development, under Waves at Entrances Work Unit 31673, Harbor Entrances and Coastal Channels Program, at the Coastal Engineering Research Center (CERC) of the US Army Engineer Waterways Experiment Station (WES). Messrs. John H. Lockhart, Jr., and John G. Housley were OCE Technical Monitors. Dr. Charles L. Vincent is CERC Program Manager.

This report was prepared by Dr. H. S. Chen, Coastal Oceanography Branch (CR-O), Research Division (CR). Work was performed under direct supervision of Dr. Edward F. Thompson, Chief, CR-O, and Mr. H. Lee Butler, Chief, CR; and under general supervision of Dr. James R. Houston and Mr. Charles C. Calhoun, Jr., Chief and Assistant Chief, CERC, respectively.

This study was initiated after a discussion by Dr. Vincent and the author on the possibility of implementing a scheme to redistribute wave energy behind islands in numerical models. The author acknowledges and appreciates the review and comments provided by Drs. Edward F. Thompson and Norman W. Scheffner. This report was edited by Ms. Shirley A. J. Hanshaw, Information Products Division, Information Technology Laboratory, WES.

Commander and Director of WES is COL Dwayne G. Lee, CE, and Technical Director is Dr. Robert W. Whalin.

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# COMBINED REFLECTION AND DIFFRACTION BY A VERTICAL WEDGE

## PART I: INTRODUCTION

1. The boundary value problem of linear wave reflection and diffraction by a vertical wedge of arbitrary wedge angle has been well formulated and presented by Stoker (1957) among many other investigators. The technique to obtain an analytical solution for the problem is also depicted in the cited book. However, analytical solutions are not available for the problem, except for the special case of wave diffraction by a thin semi-infinite breakwater, that is, a wedge with wedge angle equal to zero.

2. The solution of the thin semi-infinite breakwater was presented in the dimensionless diffraction diagrams by Wiegel (1962). The diagrams have been especially useful in preliminary engineering design and have been included in the Shore Protection Manual (SPM) (1984). Although equally useful, the combined reflection and diffraction diagrams are not available, perhaps because of the complexity of the diagrams which makes them difficult to create without using modern high-speed computers for computation and graphing.

3. The objectives of the present study are (a) to obtain an analytical solution for the combined wave reflection and diffraction by a vertical wedge of arbitrary wedge angle subject to excitation of a plane simple harmonic wave train coming from infinity and (b) to provide the combined reflection and diffraction diagrams. The diagrams included in this report have two cases: one for a thin semi-infinite breakwater and the other for a 90-deg vertical wedge. Subroutine WEDGE for computing the combined reflection and diffraction by a vertical wedge of arbitrary wedge angle is also documented in the report (Appendix A).

## PART II: BOUNDARY VALUE PROBLEM

### Mathematical Formulation

4. In this study our primary interest is the wave reflection and diffraction by a vertical wedge of arbitrary wedge angle in a constant water depth $h$ * subject to the excitation of monochromatic incident waves of infinitesimal amplitude coming from infinity. Let $(r,\theta,z)$ be cylindrical coordinates, with $z = 0$ representing the undisturbed water free surface and upward direction representing the positive z-axis. The tip of the wedge is chosen to be the origin of the coordinates and two rigid walls of the wedge to coincide with $\theta = 0$ and $\theta = \theta_o$, respectively, as illustrated in Figure 1. Cartesian coordinates $(x,y,z)$, corresponding to the cylindrical coordinates, are also occasionally used and shown in the same figure. Therefore, the wedge

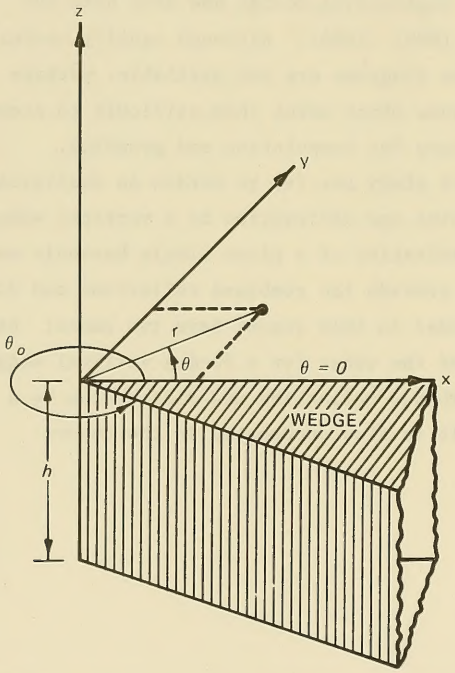


Figure 1. A vertical wedge of arbitrary wedge angle

* For convenience, symbols and abbreviations are listed in the notation (Appendix B).

angle is $2\pi - \theta_o$, and the water region is defined by $\theta_o \geq \theta \geq 0$ and $0 \geq z \geq -h$.

5. The velocity field for the wave reflection and diffraction in an ideal fluid can be represented by the velocity potential function $\Phi(r,\theta,z,t)$ which must satisfy the Laplace equation, where $t$ is the temporal coordinate. We assume that the waves are sinusoidal in time with radian frequency $\omega$. Water depth is constant, and the bottom is rigid and impermeable. Therefore, the vertical and temporal components of the velocity potential function, which follow from separation of variables, can be factored out and the velocity potential written as

$$\Phi(r,\theta,z,t) = A_o \frac{\cosh k(z + h)}{\cosh kh} \phi(r,\theta)e^{i\omega t} \tag{1}$$

where

$A_o = -iga_o/\omega$

$i = \sqrt{-1}$

$g$ = gravitational acceleration

$a_o$ = incident wave amplitude

$k$ = wave number

$\phi$ = horizontal component of the velocity potential function

6. Substituting Equation 1 into the Laplace equation and using both the kinematic and dynamic boundary conditions at the free surface, the Laplace equation is then reduced to the Helmholtz equation which is written in polar coordinates as follows:

$$r^2 \frac{\partial^2\phi}{\partial r^2} + r \frac{\partial\phi}{\partial r} + \frac{\partial^2\phi}{\partial\theta^2} + k^2r^2\phi = 0 \tag{2}$$

where $k$ must be a real number and satisfy the dispersion relationship

$$\omega^2 = gk \tanh kh \tag{3}$$

7. The free surface displacement $\eta$ from the mean water level $z = 0$ can be obtained from linear wave theory and is represented as

$$\eta(r,\theta,t) = \frac{1}{g} \frac{\partial\Phi}{\partial t} = a_o\phi(r,\theta)e^{i\omega t} \tag{4}$$

8. Thus only the horizontal part of the velocity potential function $\phi$ is needed to be determined as a solution of Equation 2 in the water region $\theta_o \geq \theta \geq 0$, with the following boundary conditions at the rigid and impermeable walls of the wedge:

$$\frac{\partial \phi}{\partial \theta} = 0 \qquad \text{at} \quad \theta = 0 \quad \text{and} \quad \theta_o \tag{5}$$

9. A condition at infinity is also required to ensure a unique solution. The classic approach is to use the Sommerfeld radiation condition at infinity which states that the scattered wave $\phi_s$ must behave like a cylindrical outgoing progressing wave at infinity such that '

$$\lim_{r \to \infty} \sqrt{r} \left( \frac{\partial \phi_s}{\partial r} + ik\phi_s \right) = 0 \tag{6}$$

The total wave represented by $\phi$ is the linear superposition of an incident wave $\phi_i$, a reflected wave from the the $\theta = 0$ wall of the wedge $\phi_r$, and the scattered wave $\phi_s$ from the tip of the wedge.

$$\phi = \phi_i + \phi_r + \phi_s \tag{7}$$

Equation 6 can be satisfied if

$$\phi_s \sim \frac{e^{-ikr}}{\sqrt{kr}} \qquad \text{at} \quad r \to \infty \tag{8}$$

10. The incident wave coming from a large distance from the tip of the wedge is assumed to be a plane progressive wave of amplitude $a_o$ and incident angle $\alpha$ to the x-axis as given by

$$\phi_i = e^{ikr \cos(\theta - \alpha)} \tag{9}$$

Consequently, the perfectly reflected wave from the $y = 0$ wall of the wedge is

$$\phi_r = e^{ikr \cos(\theta + \alpha)} \tag{10}$$

Thus the boundary value problem (in which the governing equation is Equation 2, the boundary condition is Equation 5, and the radiation condition is Equation 6) is completely formulated.

## Analytical Solution

11. Analytical solution to the problem formulated in the preceding section is obtained by following the solution technique by Stoker (1957). To obtain the solution, the water region is divided into three subregions--I, II, and III--by the incident wave ray passing through the tip of the wedge and the reflected wave ray reflected away from the tip of the wedge, as shown in Figure 2. Obviously, the total wave in subregion I is the sum of the incident, reflected, and scattered waves; the total wave in subregion II, where the reflected wave does not exist, is the sum of the incident and scattered waves; and the total wave in subregion III, where the incident and reflected waves have been shaded out, is only the scattered wave. For certain combinations of

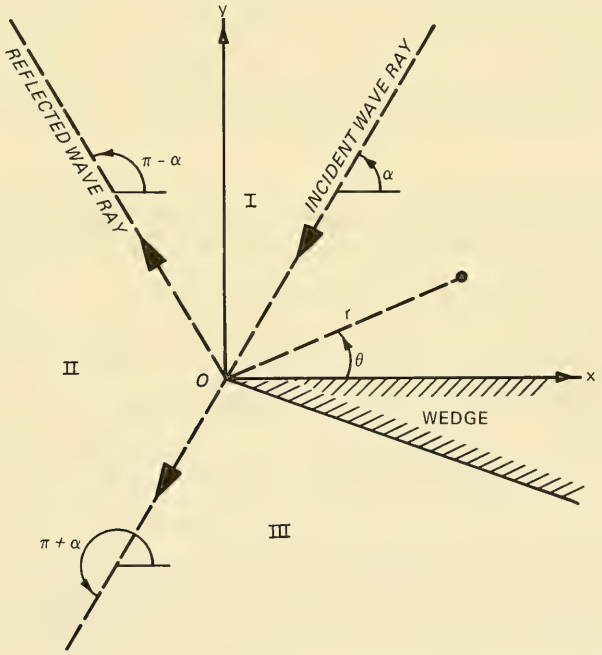


Figure 2. Three subregions and the wedge

the wedge angle and incident wave angle, subregions II and III may not exist at all. In general, the solution function can be written as

$$\phi = \phi_o(r,\theta) + \phi_s(r,\theta) \tag{11}$$

where

$$\phi_o(r,\theta) = \begin{cases} \phi_i + \phi_r & \pi - \alpha > \theta > 0 \\ \phi_i & \pi + \alpha > \theta > \pi - \alpha \\ 0 & \theta_o > \theta > \pi + \alpha \end{cases} \tag{12}$$

The equation reveals that $\phi_o$ is the sum of the incident and reflected waves $\phi_i$ and $\phi_r$ and is a known function. The scattered wave $\phi_s$ is the only unknown function to be determined in the problem. Nevertheless, the total wave $\phi$ instead of the scattered wave $\phi_s$ is the desired solution to be obtained in this study.

12. The solution for the total wave $\phi$ is pursued. The finite cosine transform of $\phi$, denoted by $\overline{\phi}$, is introduced by the formula

$$\overline{\phi}(kr,n) = \int_0^{\nu\pi} \phi(kr,\theta) \cos \frac{n\theta}{\nu} \, d\theta \tag{13}$$

where $n = 0,1,2,\ldots$ are integers, and $\nu$ is related to the wedge angle as defined by

$$\theta_o = \nu\pi \tag{14}$$

Applying the finite cosine transform and using the boundary condition in Equation 5, Equation 2 becomes

$$r^2 \frac{\partial^2 \overline{\phi}}{\partial r^2} + r \frac{\partial \overline{\phi}}{\partial r} + \left[ (kr)^2 - \left(\frac{n}{\nu}\right)^2 \right] \overline{\phi} = 0 \tag{15}$$

Equation 15 is a form of the Bessel equation for which general solutions are the Bessel functions of the first and second kinds, $J_{n/\nu}(kr)$ and $Y_{n/\nu}(kr)$, respectively. Since $Y_{n/\nu}(kr)$ are singular at the origin, the solution is chosen to be

$$\bar{\phi}(kr,n) = a_n J_{n/\nu}(kr) \tag{16}$$

where $a_n$ are constants to be determined.

13. Taking the finite cosine transform of Equation 11 and using Equation 16, we have

$$\int_0^{\nu\pi} \phi_s \cos \frac{n\theta}{\nu} \, d\theta = a_n J_{n/\nu}(kr) - \int_0^{\nu\pi} \phi_o \cos \frac{n\theta}{\nu} \, d\theta \tag{17}$$

or

$$\bar{\phi}_s = a_n J_{n/\nu}(kr) - \bar{\phi}_o \tag{18}$$

Then applying the operation $\lim_{r\to\infty} \sqrt{r}(\partial/\partial r + ik)$ to both sides of Equation 18, and using the Sommerfeld radiation condition (Equation 6) we have

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial}{\partial r} + ik\right)\left[a_n J_{n/\nu}(kr) - \int_0^{\nu\pi} \phi_o \cos \frac{n\theta}{\nu} \, d\theta\right] = 0 \tag{19}$$

14. Equation 19 can be asymptotically evaluated to determine $a_n$. Firstly, the first term involving the Bessel function is evaluated. The function $J_{n/\nu}(kr)$ at $r\to\infty$ behaves asymptotically (Abramowitz and Stegun 1964) as follows:

$$J_{n/\nu}(kr) \sim \sqrt{\frac{2}{\pi kr}} \cos\left(kr - \frac{n\pi}{2\nu} - \frac{\pi}{4}\right) \tag{20}$$

Hence, we have

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial}{\partial r} + ik\right) J_{n/\nu}(kr) \sim \sqrt{\frac{2k}{\pi}} e^{i(kr - n\pi/2\nu + \pi/4)} \tag{21}$$

Secondly, the second term involving the integral of $\phi_o$ is evaluated. The asymptotic behavior of the integral over $\theta = (0,\nu\pi)$ and at large distance $r\to\infty$ can be found by the method of stationary phase. The integral, after substituting $\phi_o$ from Equations 9, 10, and 12, can be written as

$$\int_0^{\nu\pi} \phi_o \cos\frac{n\theta}{\nu}\, d\theta = \int_0^{\pi-\alpha} \left[e^{ikr\cos(\theta-\alpha)} + e^{ikr\cos(\theta+\alpha)}\right] \cos\frac{n\theta}{\nu}\, d\theta$$

$$+ \int_{\pi-\alpha}^{\pi+\alpha} e^{ikr\cos(\theta-\alpha)} \cos\frac{n\theta}{\nu}\, d\theta \qquad (22)$$

In the integrals, there are three points of stationary phase at $\theta = \alpha$ and $\theta = \pi \pm \alpha$. If the same argument as that of Stoker (1957) is followed, of the three contributions only the first one $\theta = \alpha$ furnishes a nonvanishing contribution for $r\to\infty$ when the operator $\sqrt{r}(\partial/\partial r + ik)$ is applied to it. The physical significance of this statement is that only the incident wave is effective in determining the cosine coefficients of the solution. Therefore,

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial}{\partial r} + ik\right) \int_0^{\nu\pi} \phi_o \cos\frac{n\theta}{\nu}\, d\theta \sim 2\sqrt{2\pi k} \cos\frac{n\alpha}{\nu}\, e^{i(kr+\pi/4)} \qquad (23)$$

Substituting Equations 21 and 23 into Equation 19, we obtain the unknown coefficients $a_n$:

$$a_n = 2\pi \cos\frac{n\alpha}{\nu}\, e^{in\pi/2\nu} \qquad (24)$$

15. Since the solution $\phi$ in the cosine series expression is

$$\phi(r,\theta) = \frac{1}{\nu\pi}\,\bar{\phi}(r,0) + \frac{2}{\nu\pi}\sum_{n=1}^{\infty} \bar{\phi}(r,n) \cos\frac{n\theta}{\nu} \qquad (25)$$

the solution is obtained by substituting Equations 16 and 24 into Equation 25 as follows:

$$\phi(r,\theta) = \frac{2}{\nu}\left[J_0(kr) + 2\sum_{n=1}^{\infty} e^{in\pi/2\nu} J_{n/\nu}(kr) \cos\frac{n\alpha}{\nu} \cos\frac{n\theta}{\nu}\right] \tag{26}$$

Equation 26 is the solution for the combined wave reflection and diffraction by a vertical wedge of arbitrary wedge angle and is considered to be extended from the solution by Stoker (1957) who only solved the problem of a thin semi-infinite breakwater. The solution in Equation 26 and the one by Stoker are not only in nonclosed form but also in terms of Bessel functions. It seems that the calculations of the solutions are very difficult without using a modern high-speed computer. This is probably the reason why Stoker arrived at his solution expressed in the same cosine series but did not use it to calculate the result. Instead, he further transformed the expression into a very complex integral form for further approximation in calculating the result.

16. Notably, the solution at the origin point is obtained by simply substituting $r = 0$ into Equation 26 to arrive at

$$\phi(0,\theta) = \frac{2}{\nu} \tag{27}$$

Therefore, wave response at the origin point depends only on the wedge angle and does not depend on the incident wave angle.

## Two Special Cases

17. The solutions for two special cases are used to verify Equation 26: one for the case of a thin semi-infinite breakwater and the other for the case of an infinite wall extending from $x = -\infty$ to $\infty$.

18. The vertical wedge should reduce to a thin semi-infinite breakwater as the wedge angle reduces to 0 deg. Therefore, solution of the combined wave reflection and diffraction by a thin semi-infinite breakwater is obtained by substituting $\nu = 2$ (that is, $\theta_o = 2\pi$) into Equation 26 which then becomes

$$\phi(r,\theta) = J_0(kr) + 2 \sum_{n=1}^{\infty} e^{in\pi/4} J_{n/2}(kr) \cos \frac{n\alpha}{2} \cos \frac{n\theta}{2} \tag{28}$$

Equation 28 is precisely the same one obtained by Stoker (1957).

19. The vertical wedge should also become an infinite wall extending from $x = -\infty$ to $\infty$ with the water occupying only the half plane of $y \geq 0$ as the wedge angle increases to 180 deg. In this situation the scattered wave is absent from the solution, and the total wave is only the sum of the incident and reflected waves as follows:

$$\phi(r,\theta) = e^{ikr \cos(\theta-\alpha)} + e^{ikr \cos(\theta+\alpha)} \tag{29}$$

After expansion of the exponential functions in terms of Bessel functions (Abramowitz and Stegun 1964), Equation 29 becomes

$$\phi(r,\theta) = 2\left[J_0(kr) + 2 \sum_{n=1}^{\infty} i^n J_n(kr) \cos n\alpha \cos n\theta\right] \tag{30}$$

Equation 30 is the same equation reduced from Equation 26 by substituting $\nu = 1$ into it.

## PART III: CALCULATION AND RESULTS

20. Results of the combined reflection and diffraction by a wedge of arbitrary wedge angle can be calculated from Equation 26. Since the solution is not only in terms of Bessel functions but also in a nonclosed form, the computer program WEDGE is therefore written to calculate the solution.

21. In the program the subroutine BESJ for calculating Bessel function of fractional or integer order was used. The subroutine was originally written by Amos, Daniel, and Weston in 1975 (Morris 1984) and is collected in the Naval Surface Weapons Center Library of Mathematics Subroutines (Morris 1984).

22. In the calculation the summation of the infinite terms in Equation 26 was carried out to the term which is preceded by eight successive terms of the absolute value of the Bessel function, all equal to or less than $10^{-8}$. The solution has a truncation error less than $10^{-8}$, and it is of the order of one.

23. In this study, results of the combined wave reflection and diffraction for the wedge are calculated for two cases: one for a vertical wedge of 0-deg wedge angle and the other for a vertical wedge of 90-deg wedge angle.

### Vertical Wedge of 0-Deg Wedge Angle

24. When the wedge angle is equal to zero, the wedge is actually a thin semi-infinite breakwater extending from $x = 0$ to $\infty$. Figure 3 shows the thin semi-infinite breakwater along with the polar coordinates. In this case the diffraction results for various incident wave angles in the water region from $\theta = \pi$ to $2\pi$ and $r/\lambda \leq 10$, where $\lambda$ is the incident wave length, have already been presented by Wiegel (1962) and are shown in the SPM, Volume I (1984). The present results combine reflection and diffraction effects and cover the water region from $\theta = 0$ to $2\pi$ and $r/\lambda \leq 10$. Therefore, the present results for this particular case can be considered to be a complementary and extended version to the ones in the SPM.

25. In this study wave response was calculated at 1,460 grid points intersected by $r/\lambda = 0.5,(0.5),10$, which means that the values of $r/\lambda$ are from 0.5 to 10.0 with each value increment being 0.5. Hereafter, all similar expressions are to be interpreted in the same way (e.g., $\theta = 0,(\pi/36),2\pi$ for

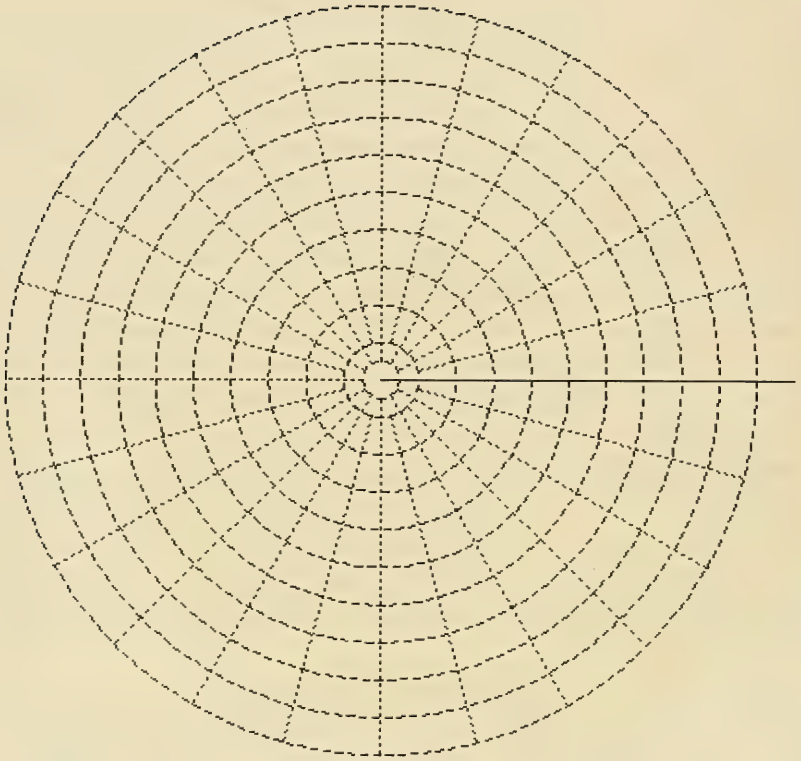

Figure 3. Thin semi-infinite breakwater and polar coordinates

the incident wave angle $\alpha = 0,(\pi/12),\pi)$ . The wave response at the origin point is obtained substituting $\nu = 2$ into Equation 27, as follows:

$$\phi(0,\theta) = 1 \tag{31}$$

Those calculated values were used to interpret the value for each non-overlapping pixel of size $0.1r/\lambda$ by $0.1r/\lambda$ in the area within the $10r/\lambda$ radius from the origin. A diagram was then constructed by patching those pixels over the entire area. The wave response diagrams for each incident wave angle are shown in Figures 4 through 15. Notably, the values in the diagrams constitute the amplification factor which is defined as the ratio of the total wave height to the incident wave height. Therefore, in subregions II

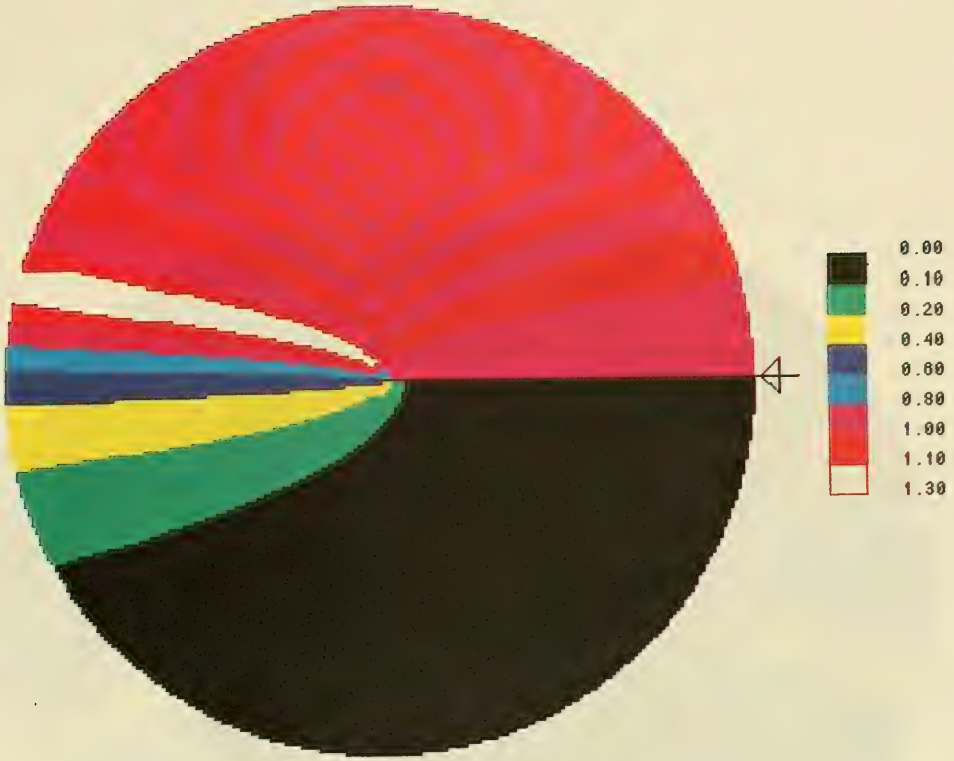


Figure 4. Amplification factor diagram for the thin semi-infinite breakwater for incident wave angle = 0 deg

and III (as defined in Figure 2) where the reflected wave is absent, the amplification factor is essentially the diffraction coefficient as defined in the SPM.

26. Figures 4 through 15 reveal that the amplification factors in subregion I change very rapidly between 0 and 2.35 over the subregion, and the diagram patterns become very complex because of the interesting superposition of the incident, reflected, and scattered waves. (In the legend of Figures 4 through 15, the width of the pixel is one incident wave length, and the values are amplification factors.) Such patterns would be very difficult to construct without using a high-speed computer and computer graphics. In subregions II and III, the amplification factors change smoothly from 1.15 roughly along the reflected wave ray reflected from the origin point to nearly

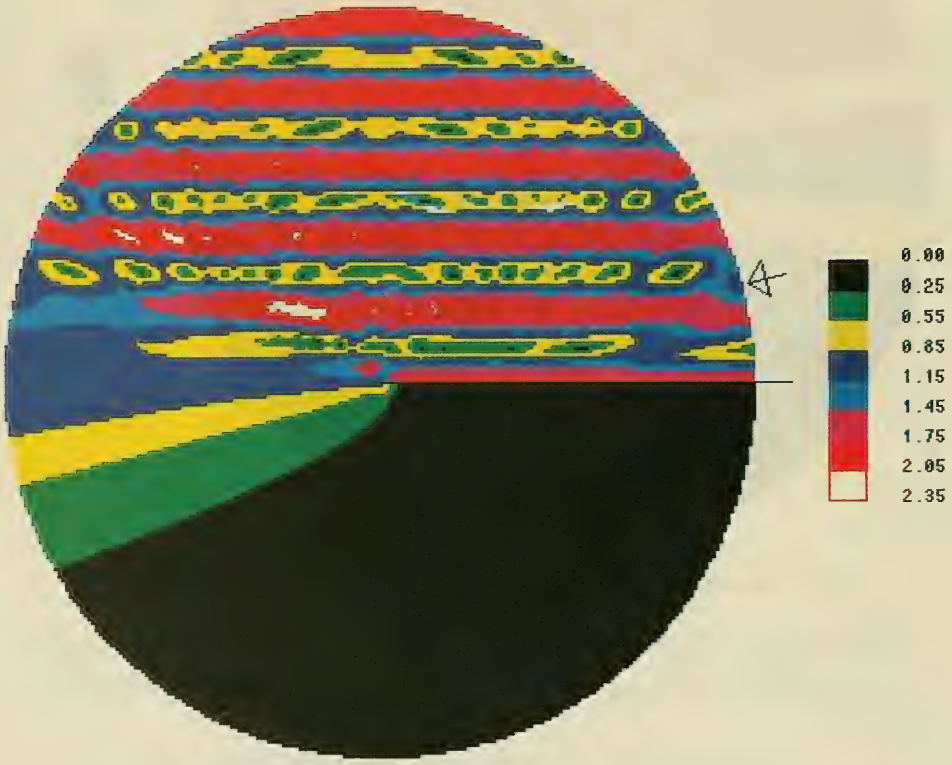


Figure 5. Amplification factor diagram for the thin semi-infinite breakwater for incident wave angle = 15 deg

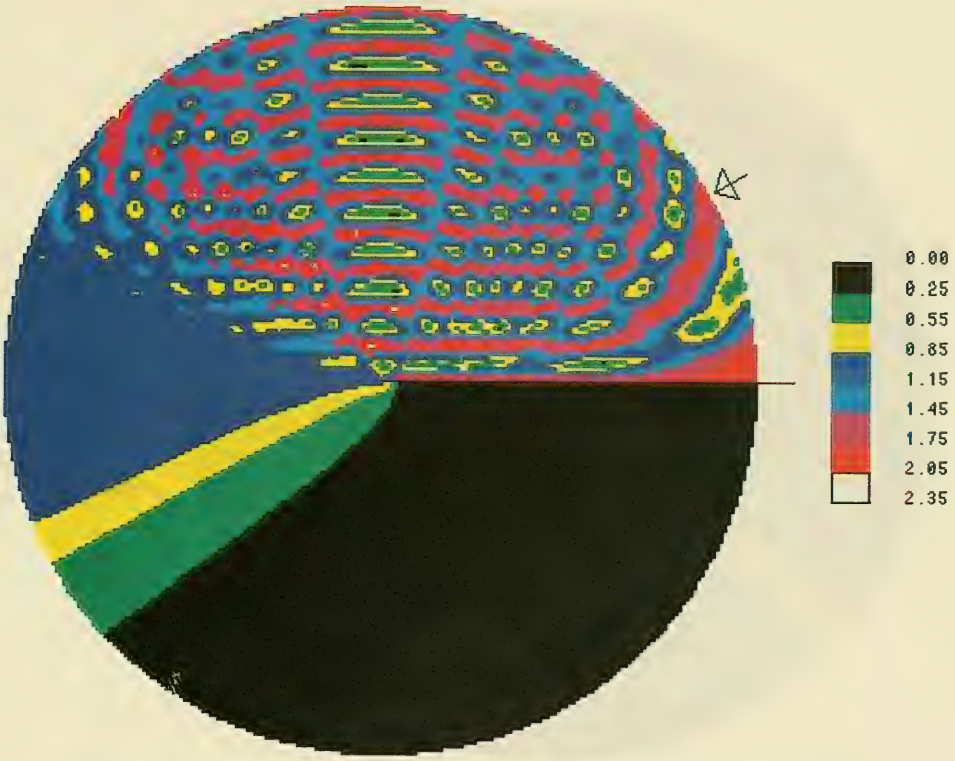


Figure 6. Amplification factor diagram for the thin semi-infinite breakwater for incident wave angle = 30 deg

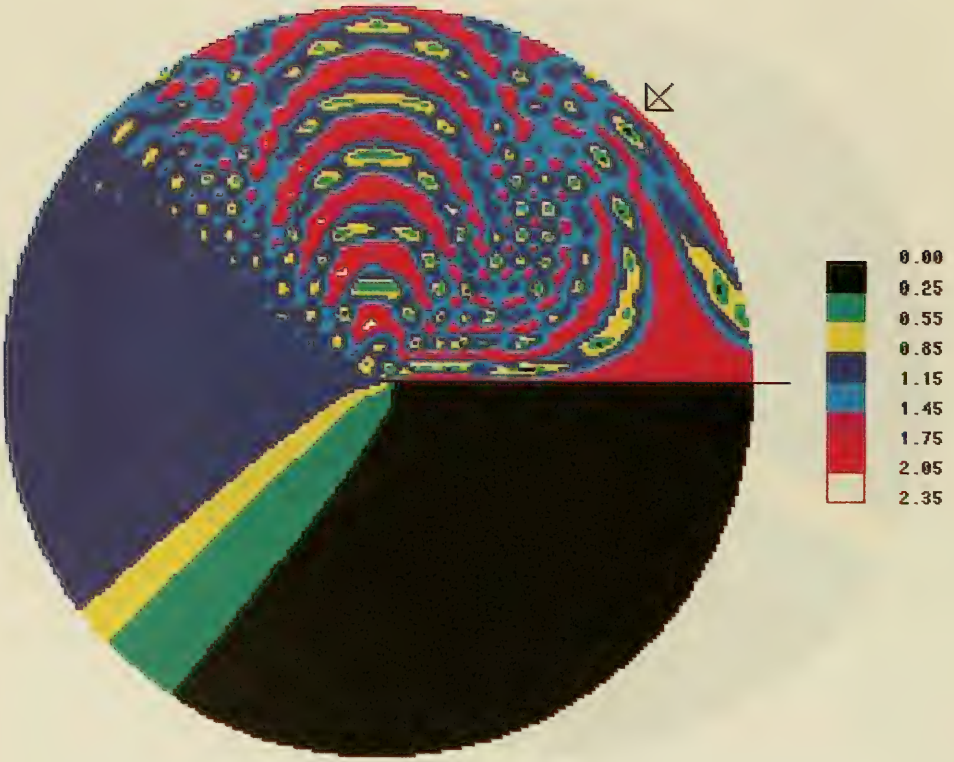


Figure 7. Amplification factor diagram for the thin semi-infinite breakwater for incident wave angle = 45 deg

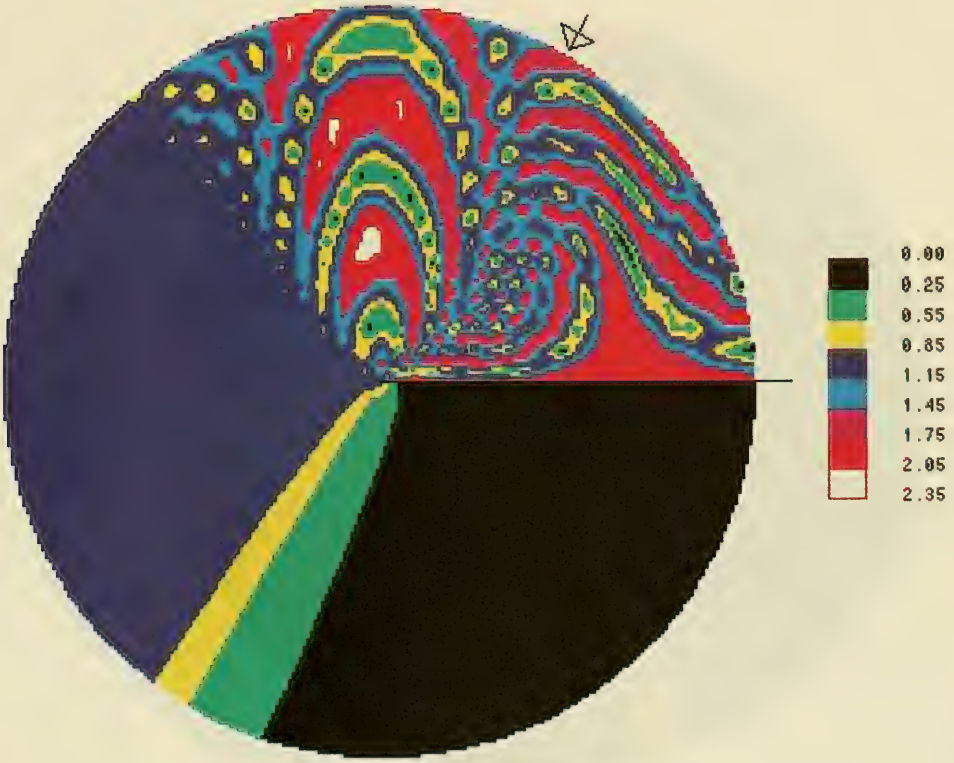


Figure 8. Amplification factor diagram for the thin semi-infinite breakwater for incident wave angle = 60 deg

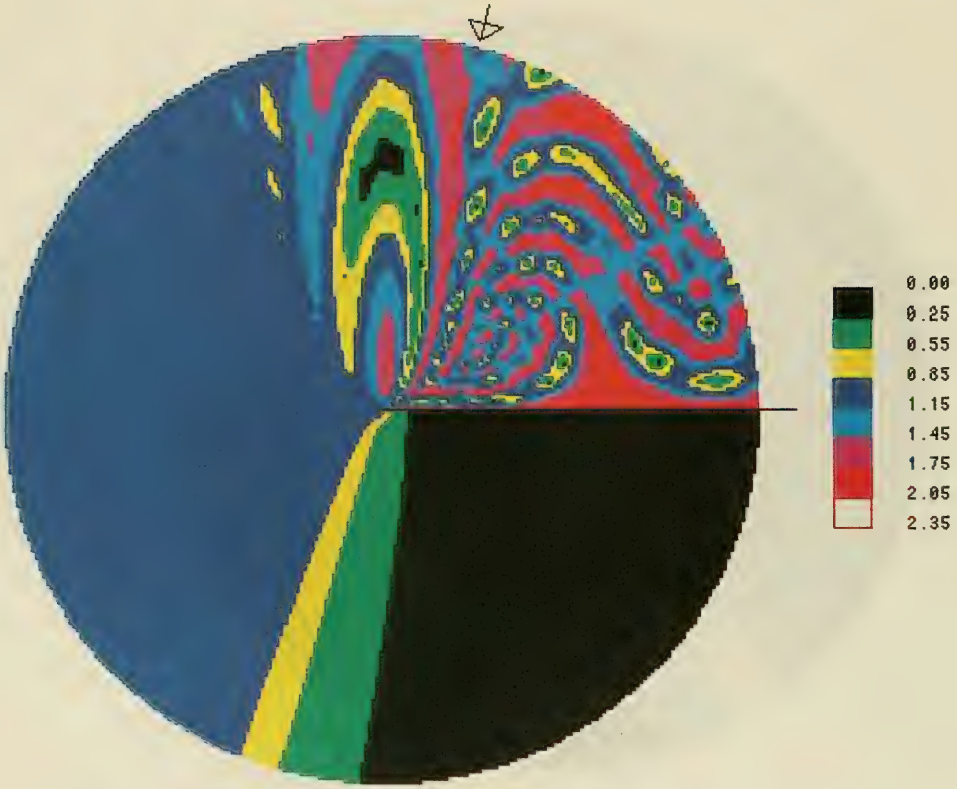


Figure 9. Amplification factor diagram for the thin semi-infinite breakwater for incident wave angle = 75 deg

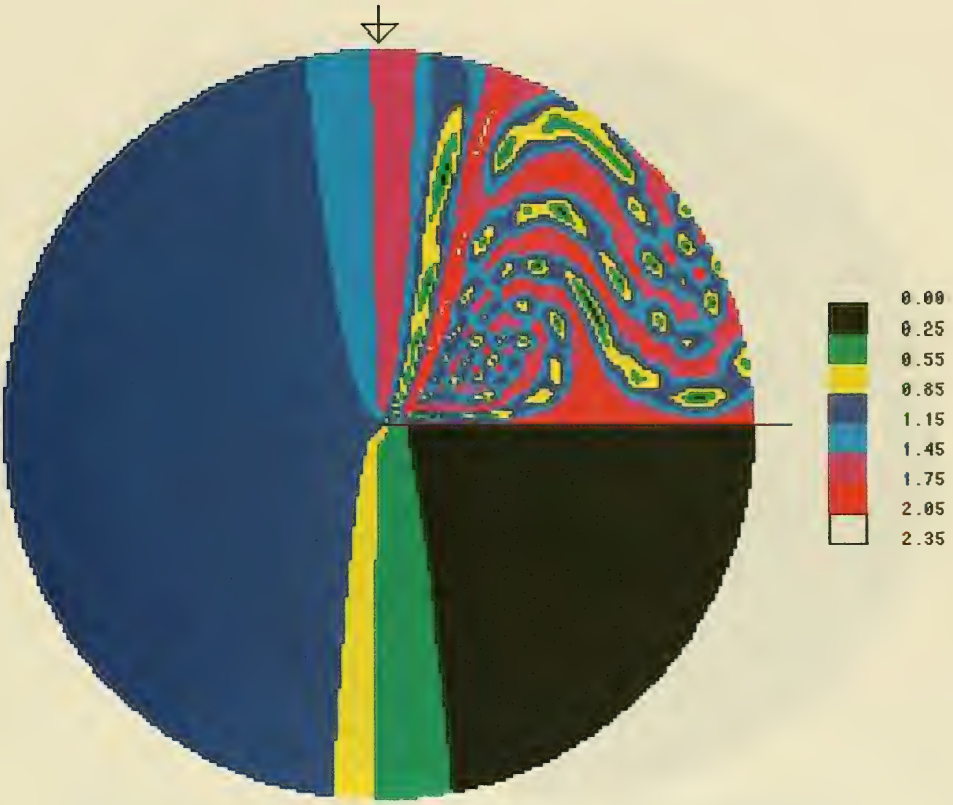


Figure 10. Amplification factor diagram for the thin semi-infinite breakwater for incident wave angle = 90 deg

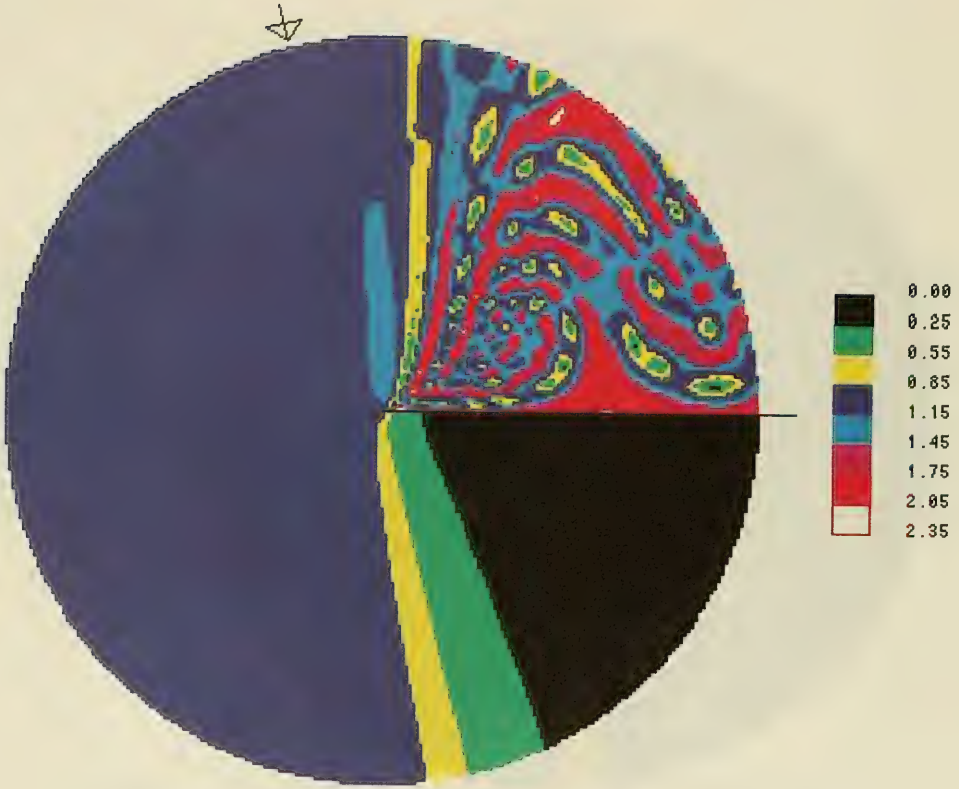


Figure 11. Amplification factor diagram for the thin semi-infinite breakwater for incident wave angle = 105 deg

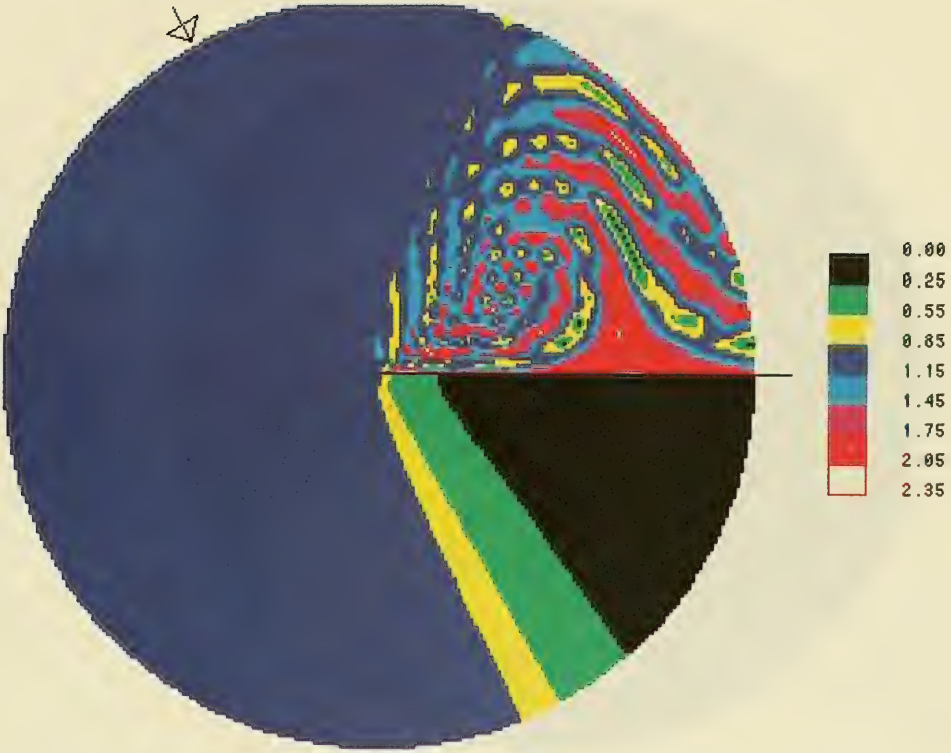


Figure 12. Amplification factor diagram for the thin semi-infinite breakwater for incident wave angle = 120 deg

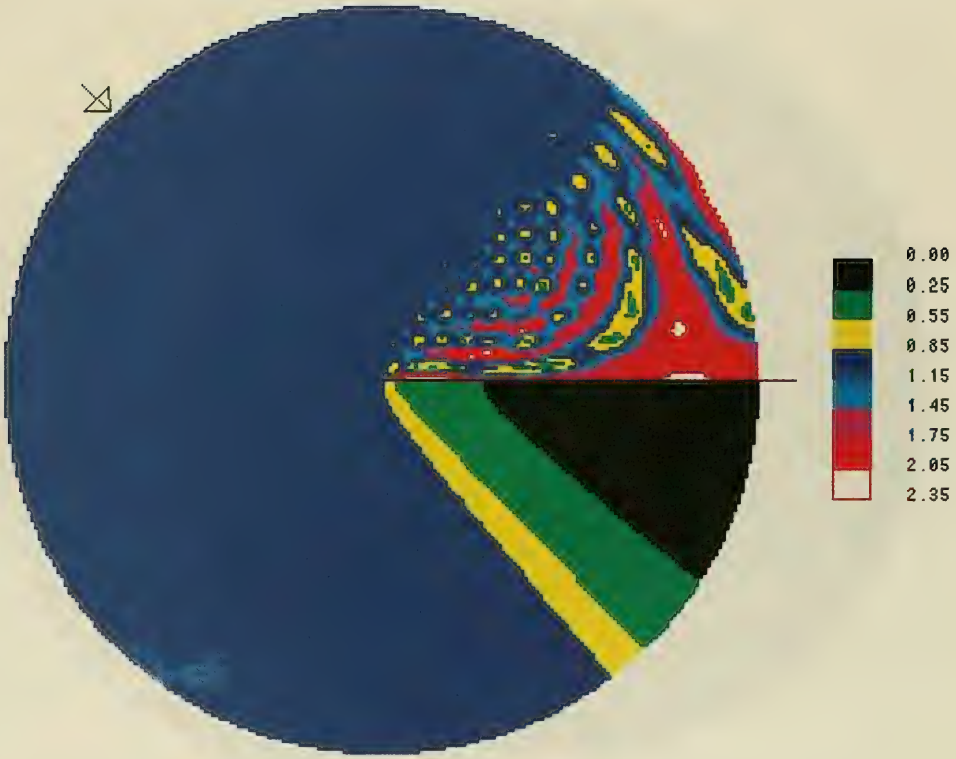


Figure 13. Amplification factor diagram for the thin semi-infinite breakwater for incident wave angle = 135 deg

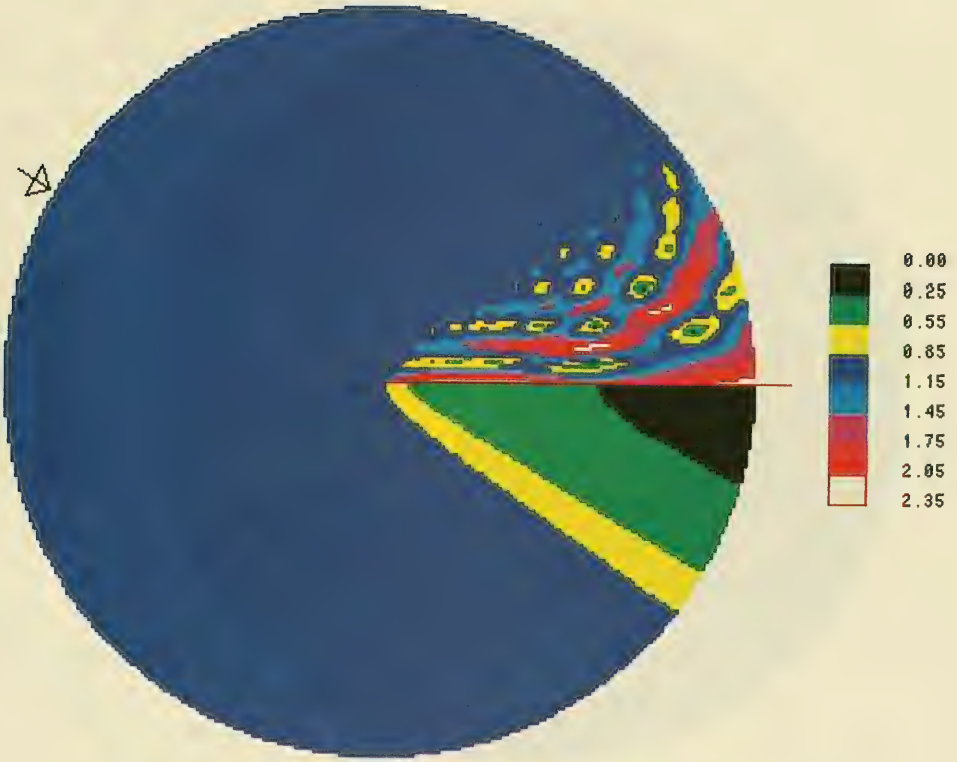


Figure 14. Amplification factor diagram for the thin semi-infinite breakwater for incident wave angle = 150 deg

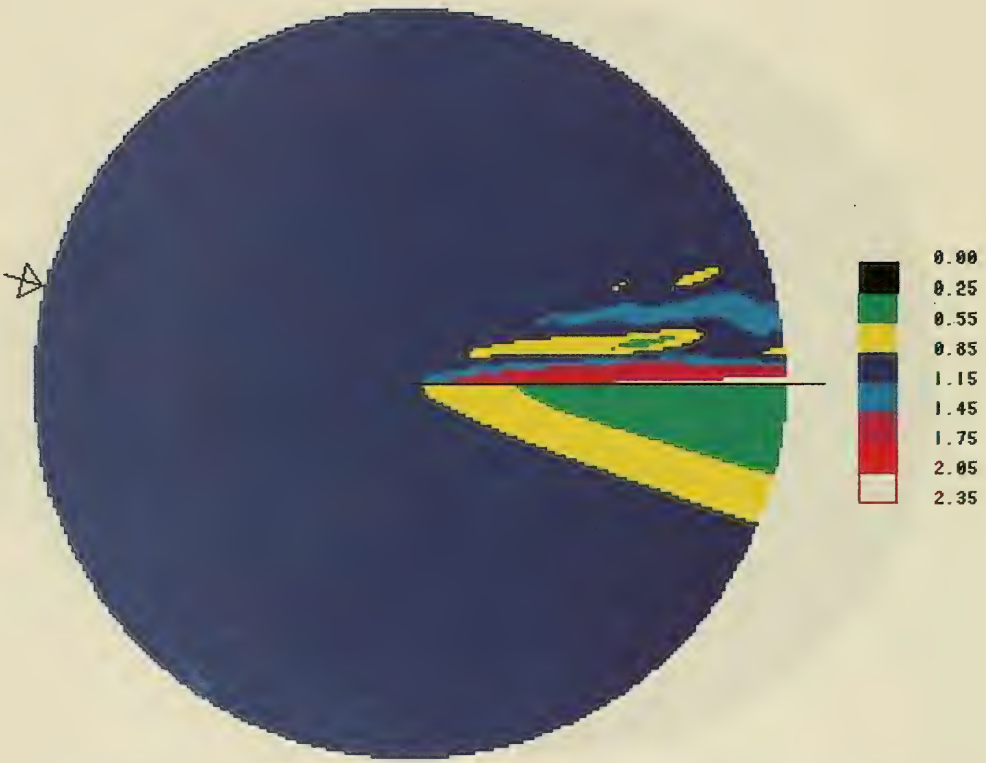


Figure 15. Amplification factor diagram for the thin semi-infinite breakwater for incident wave angle = 165 deg

0.00 at the back wall of the wedge in the shadow zone. The diagram patterns are relatively smooth and simple.

27. Notably, the diagrams do not include phase information of the wave response which is usually unimportant in most engineering practice. Should the phase of the wave response need to be known, one can always use the computer program WEDGE to calculate it.

28. The contour diagrams of the amplification factor, similar to the ones presented by Wiegel (1962) and shown in the SPM (1984), were also plotted as typically shown in Figure 16. Examination of those contour diagrams indicates that, in subregion III, the results are identical to Wiegel's results. But in subregion II the contour patterns for the amplification factor (or the diffraction coefficient $K'$ used in the SPM (1984)) equal to 1.0; thus the present results are far more complicated than Wiegel's. The author believes that Wiegel's results may lose accuracy because of insufficient resolution of the computational tools during the late fifties and early sixties. Nevertheless, such inaccuracies are usually either tolerable or immaterial in most engineering practice.

29. The contour patterns in subregion I are very complex, and it is difficult to track specific contours. Therefore, for clarity only the patched diagrams are presented, and the contour diagrams are omitted in this report.

## Vertical Wedge of 90-Deg Wedge Angle

30. When the wedge angle is equal to $\pi/2(\theta_o = 3\pi/2)$, the vertical wedge occupies the entire fourth quadrant of the space as shown in Figure 17. Wave response was calculated at 1,100 grid points intersected at $r/\lambda = 0.5,(0.5),10.0$ and $\theta = 0,(\pi/36),3\pi/2$ for the incident wave angle $\alpha = 0,(\pi/12),\pi$. The wave response at the origin is obtained by substituting $\nu = 1.5$ into Equation 27 as follows:

$$\phi(0,\theta) = \frac{4}{3} \tag{32}$$

Those calculated results were used to construct the amplification factor diagrams by following the same procedures described for the case of the thin semi-infinite breakwater. The diagrams are shown in Figures 18 through 27. Because of symmetry of the results, the diagrams for the incident wave angle $\alpha > 3\pi/4$ can be obtained from those for an incident wave angle of $\pi - \alpha$.

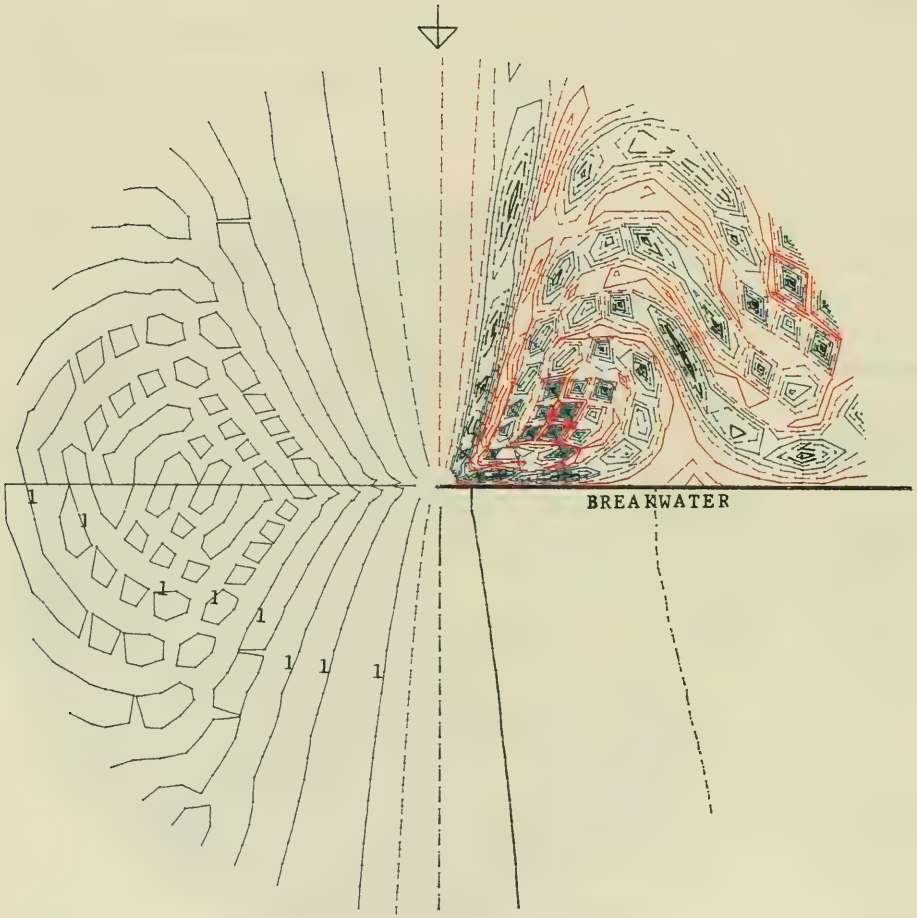


Figure 16. Amplification factor contour diagram for the thin semi-infinite breakwater for incident wave angle = 90 deg

Therefore, the diagrams for the incident wave angles $\alpha = \pi/6, (\pi/12), \pi$ are omitted in this report.

31. The diagrams indicate that, for each corresponding incident wave angle, the results in subregion I are very similar to those obtained for the vertical wedge of 0-deg wedge angle. However, the results in subregions II and III from both wedges are discernibly different.

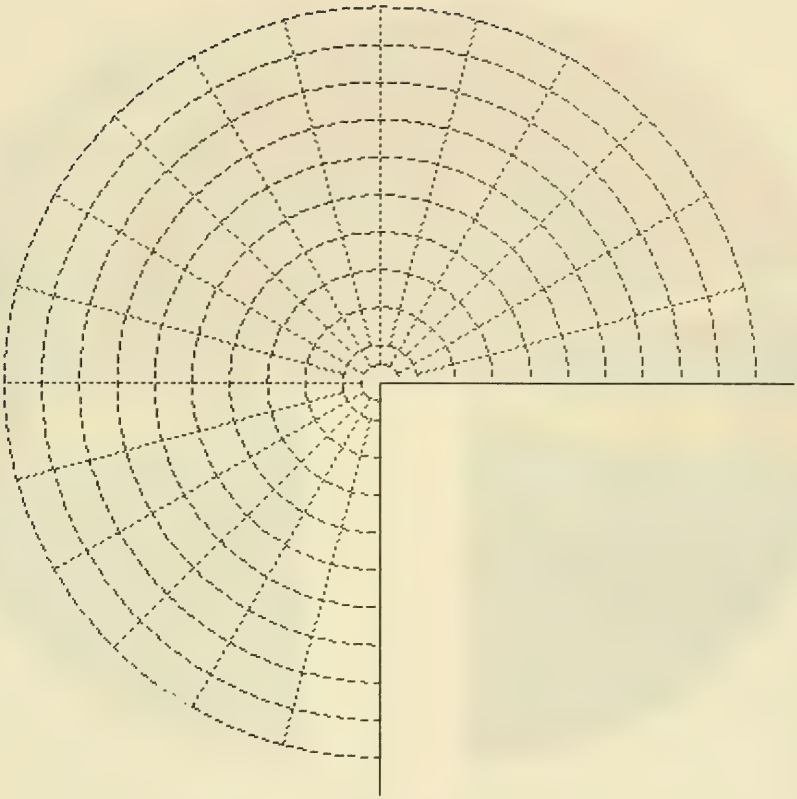

Figure 17. A 90-deg wedge and polar coordinates

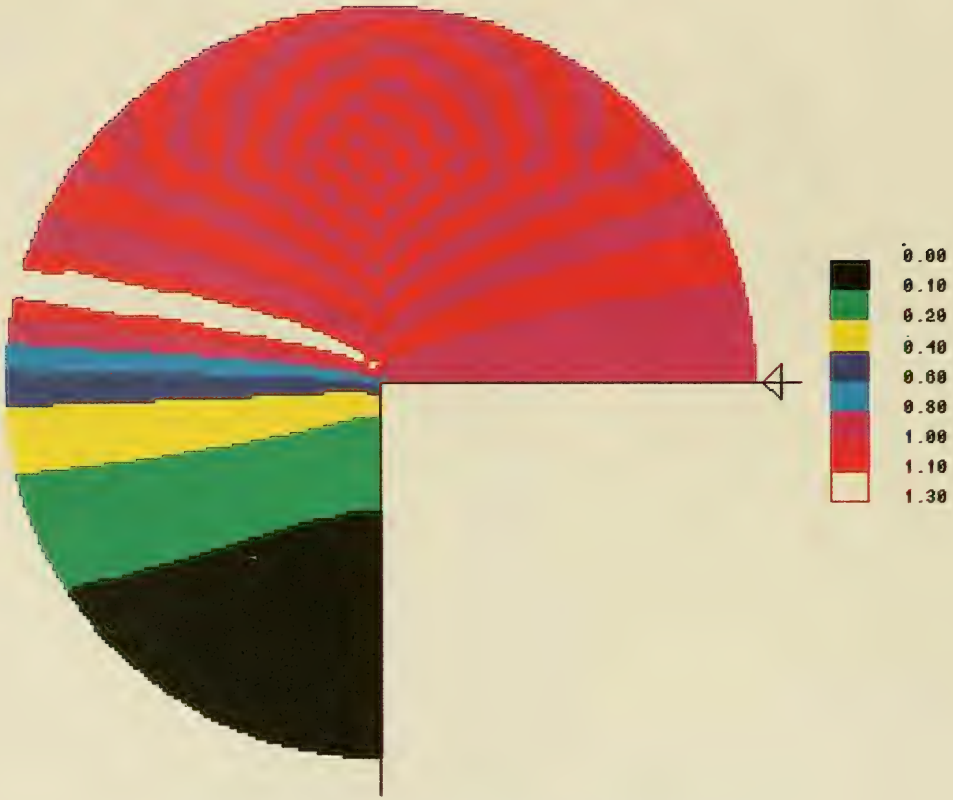


Figure 18. Amplification factor diagram for the 90-deg wedge for incident wave angle = 0 deg

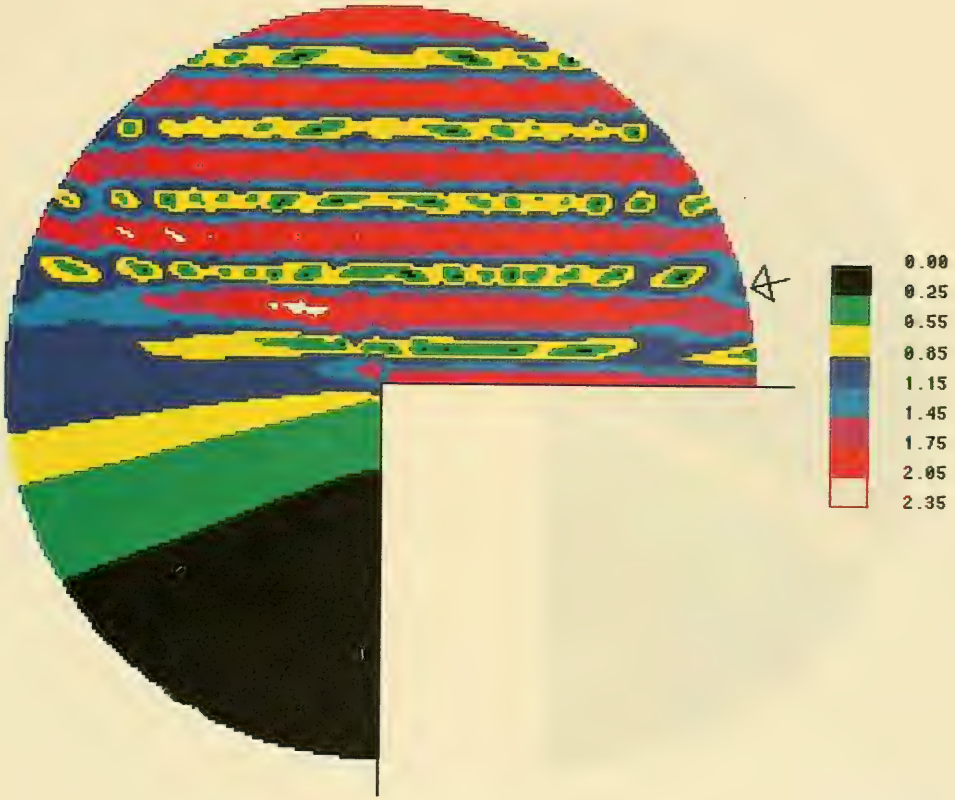


Figure 19. Amplification factor diagram for the 90-deg wedge for incident wave angle = 15 deg

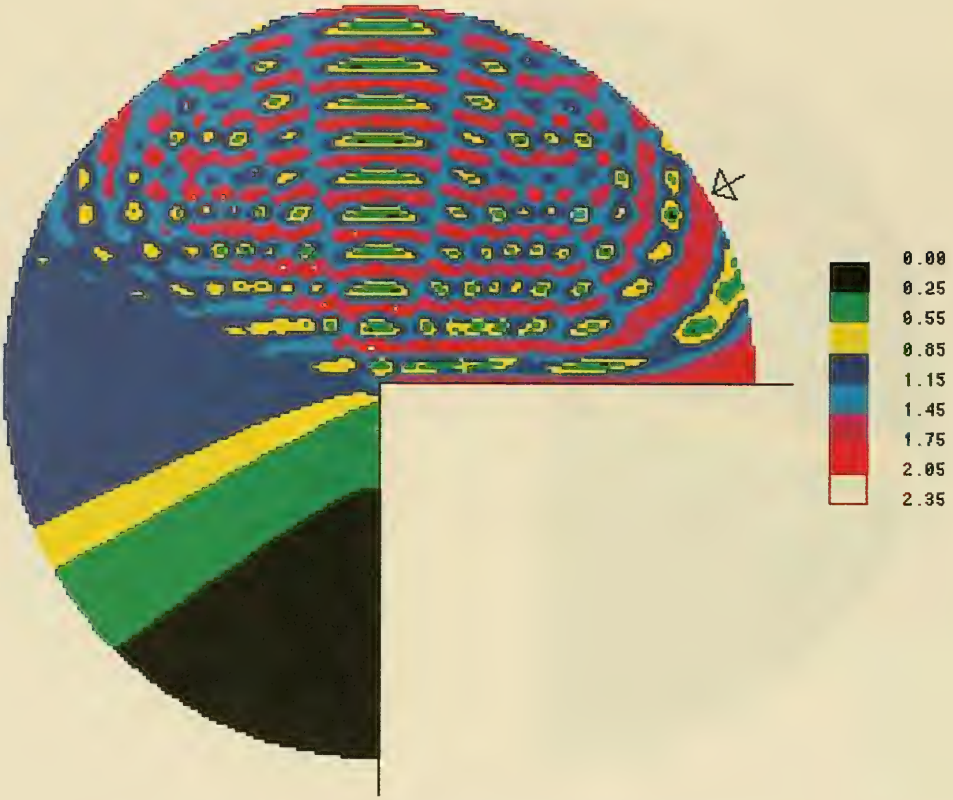


Figure 20. Amplification factor diagram for the 90-deg wedge for incident wave angle = 30 deg

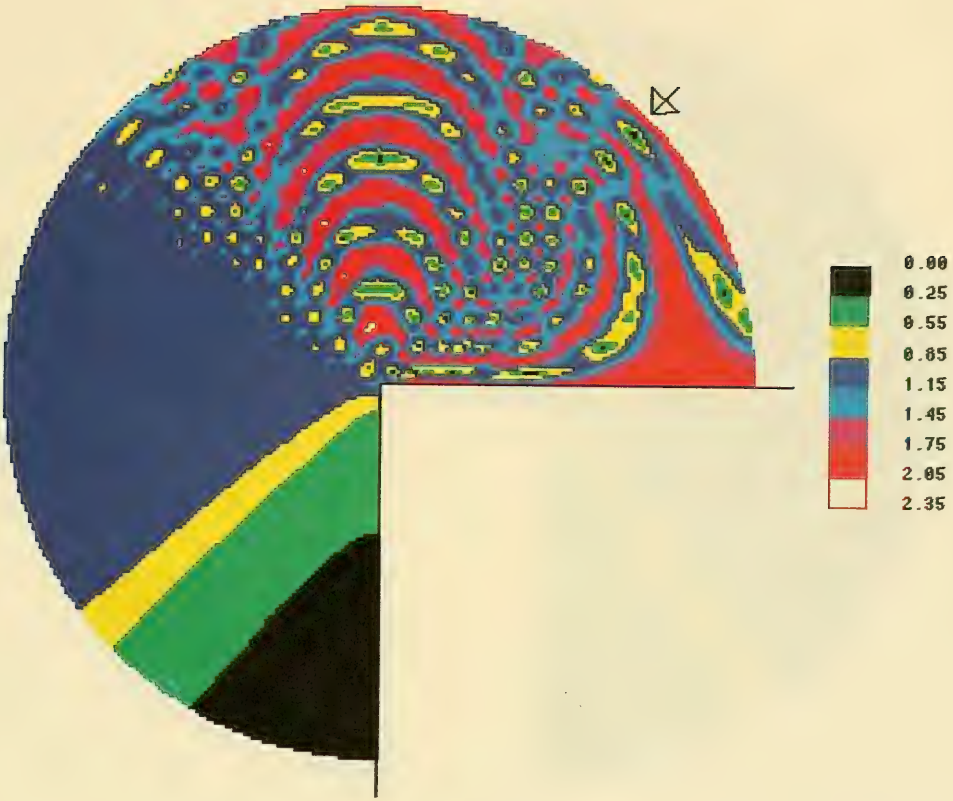


Figure 21. Amplification factor diagram for the 90-deg wedge for incident wave angle = 45 deg

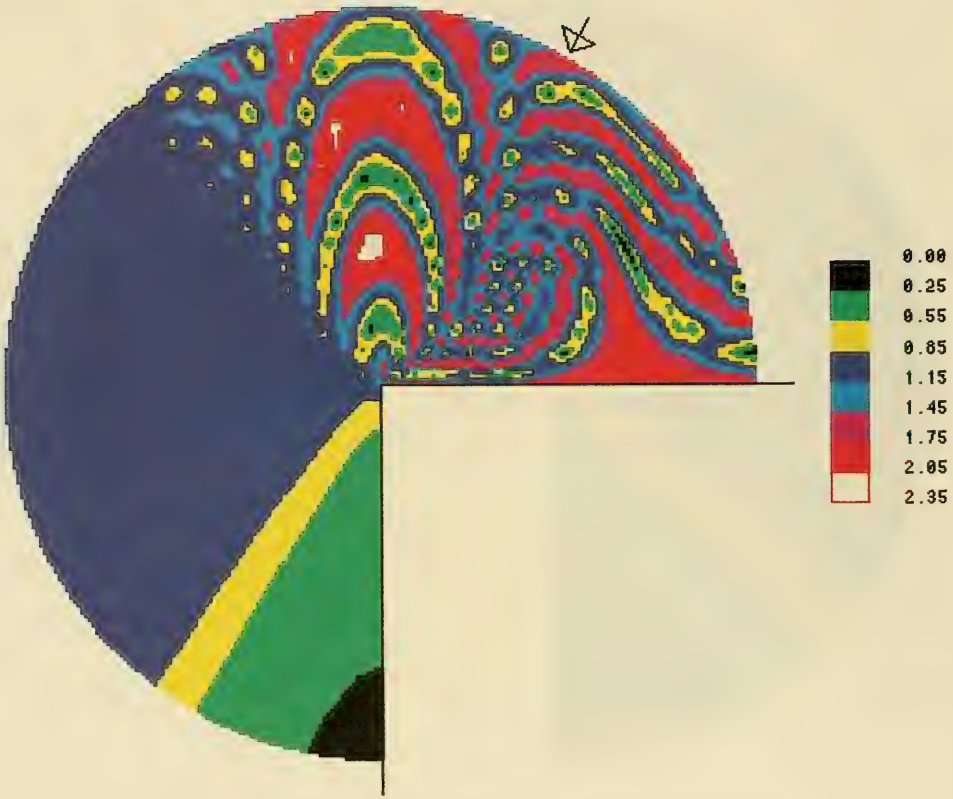


Figure 22. Amplification factor diagram for the 90-deg wedge for incident wave angle = 60 deg

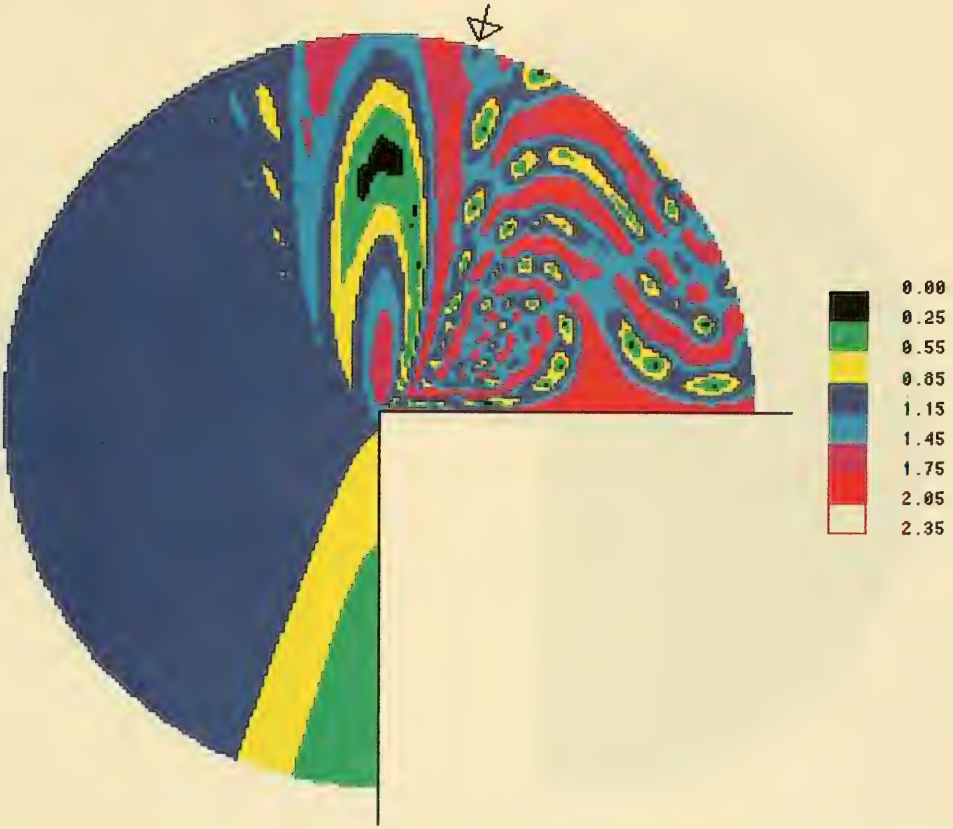


Figure 23. Amplification factor diagram for the 90-deg wedge for incident wave angle = 75 deg

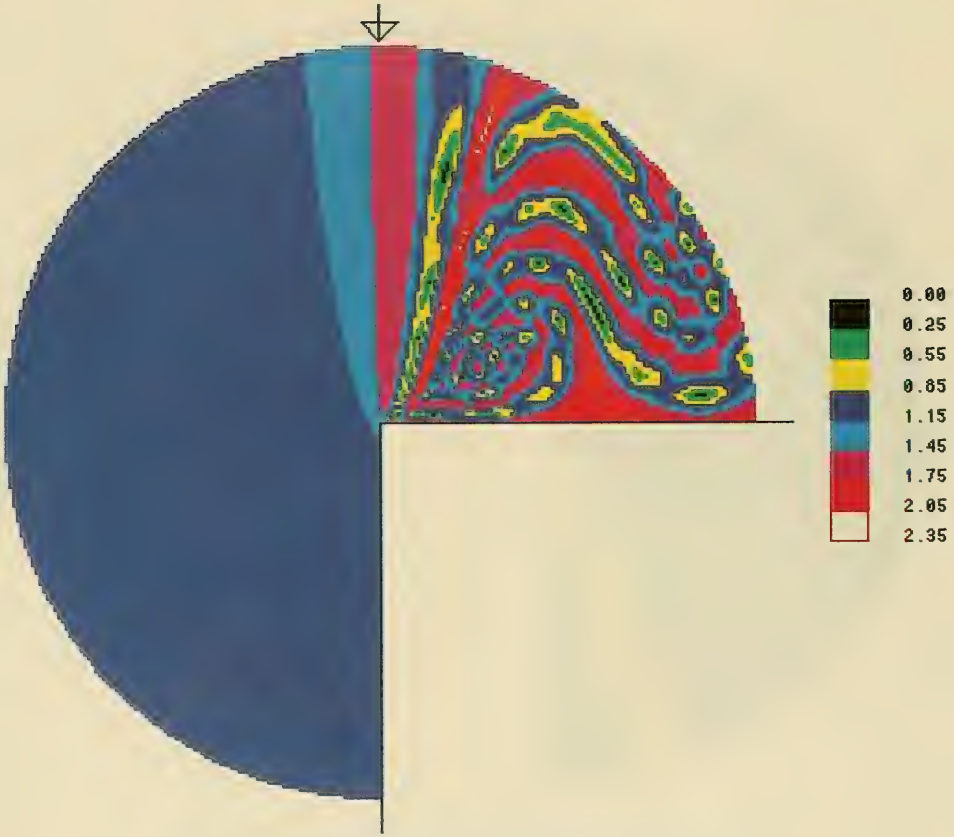


Figure 24. Amplification factor diagram for the 90-deg wedge for incident wave angle = 90 deg

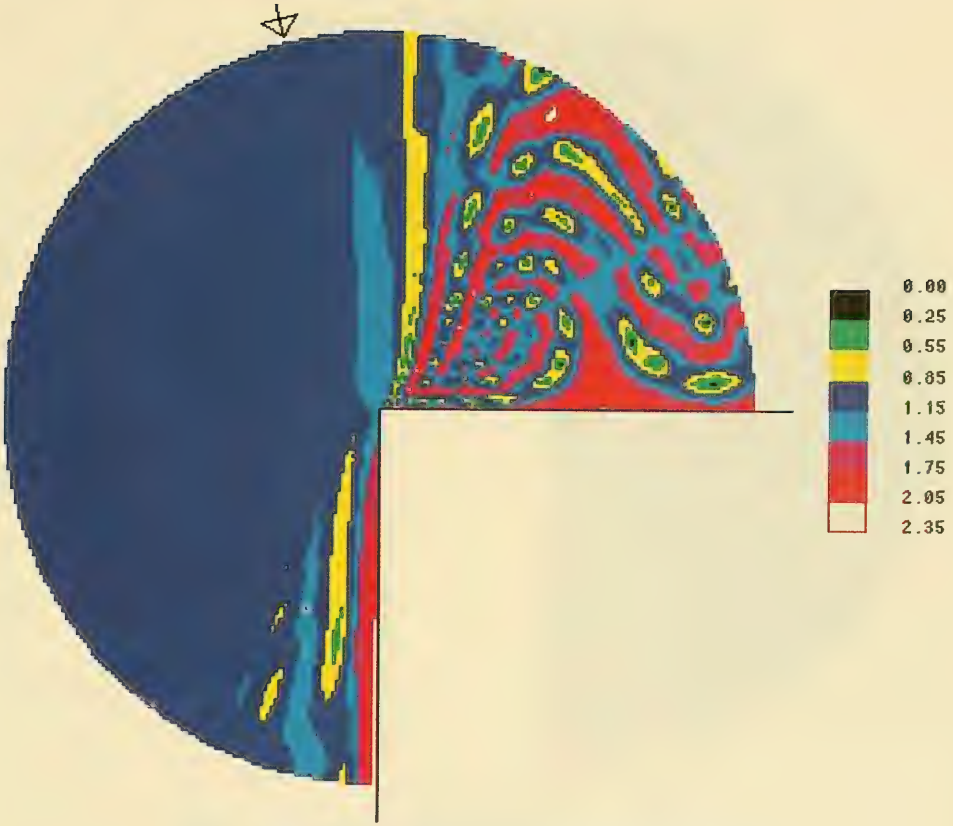


Figure 25. Amplification factor diagram for the 90-deg wedge for incident wave angle = 105 deg

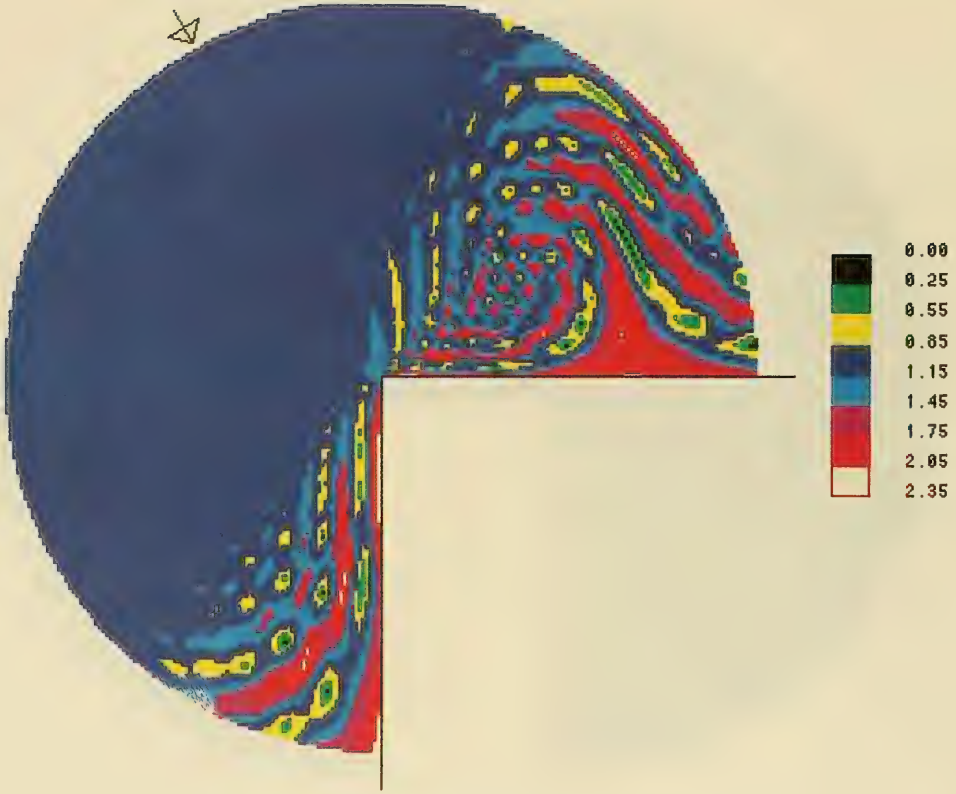


Figure 26. Amplification factor diagram for the 90-deg wedge for incident wave angle = 120 deg

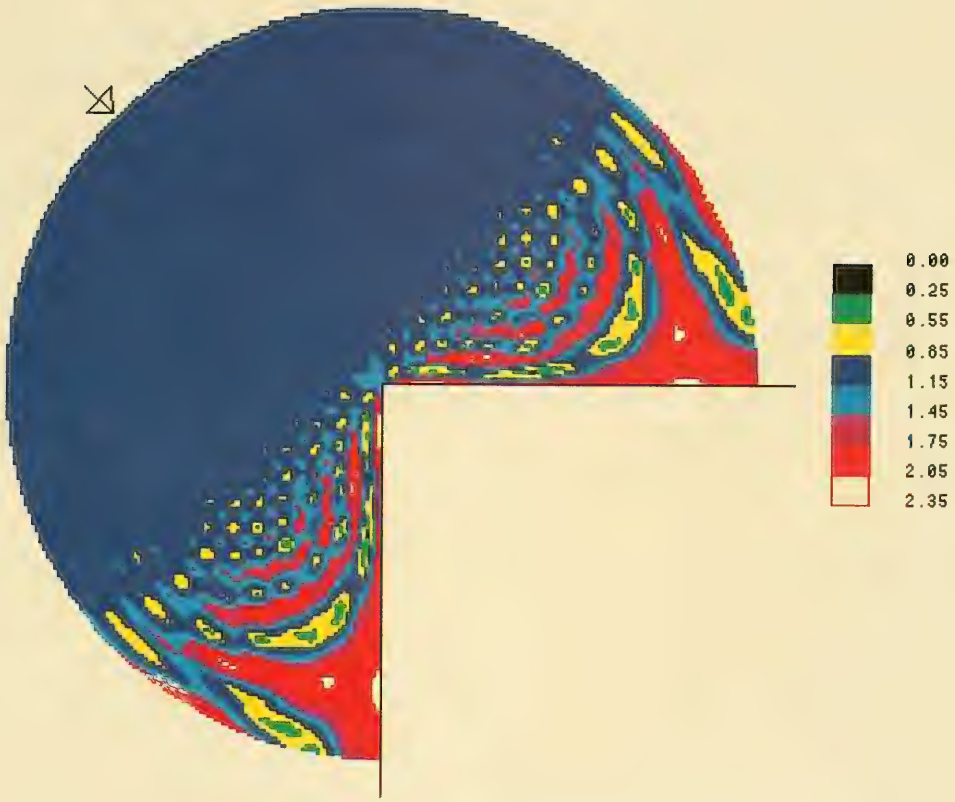


Figure 27. Amplification factor diagram for the 90-deg wedge for incident wave angle = 135 deg

## PART IV: CONCLUSION

32. An analytical solution for the combined wave reflection and diffraction by a vertical wedge of arbitrary wedge angle is obtained and expressed in Equation 26. The analytical solution is in terms of Bessel functions and in nonclosed form. The computer subroutine WEDGE, written for calculating the solution, is documented in Appendix A.

33. The amplification factor diagrams for a vertical wedge of 0-deg wedge angle and a vertical wedge of 90-deg wedge angle are calculated and presented. The calculated results indicate that the wave response in subregion I, where the incident, reflected, and scattered waves all exist, is in a very complex pattern with the amplification factor varying from 2.35 to 0.0 over the subregion. The wave response in subregions II and III is a relatively simple pattern with the amplification factor decreasing from 1.15 roughly along the reflected wave ray reflected from the origin point to nearly 0.00 at the back wall of the wedge in the shadow zone.

34. Diagrams of the special case of a vertical wedge of 0-deg wedge angle can be considered complementary and extended versions to the ones presented in the SPM (1984).

## REFERENCES


Abramowitz, M., and Stegun, I. A. 1964 (Jun). Handbook of Mathematical Functions, National Bureau of Standards, Applied Mathematics Series 55, pp 358-495.

Morris, A. H., Jr. 1984 (Jun). "NSWC Library of Mathematics Subroutines," NSWC TR 84-143, Strategic Systems Department, Naval Surface Weapons Center, Dahlgren, Va., pp 43-44.

Shore Protection Manual. 1984. 4th ed., 2 vols, US Army Engineer Waterways Experiment Station, Coastal Engineering Research Center, US Government Printing Office, Washington, DC.

Stoker, J. J. 1957. Water Waves, Interscience Publishers, Inc., New York, pp 109-133.

Wiegel, R. L. 1962. (Jan). "Diffraction of Waves by a Semi-infinite Breaker," Journal of the Hydraulics Division, American Society of Civil Engineers, Vol 88, No. HY1, pp 27-44.

## APPENDIX A: SUBROUTINE WEDGE

1. Subroutine WEDGE is used to calculate the value of $\phi$ in Equation 26 which is generally a complex number. Its absolute value is the amplification factor, and its phase is the phase indicator from the phase of the incident wave. As mentioned in the main text, $\phi$ is a function of the Bessel function of either fractional or integer order, depending on the wedge angle, and is the summation of a series of infinite terms. The subroutine BESJ, documented in the Naval Surface Weapons Center (NSWC) Library of Mathematics Subroutines (Morris 1984)* is used in the WEDGE subroutine. The programming of the WEDGE subroutine is very straightforward if a truncation term in the series in Equation 26 is determined. The program is written in FORTRAN language and is listed in this appendix.

### Description

2. The following subroutine is available for computing $\phi$ in Equation 26:

```
CALL WEDGE(F,FABS,FPHA,XRL,XTH,WEDGEA,WAVEA,IDX)
```

where the arguments are all real values except F which is a complex value.

Input arguments are as follows:

a. (XRL,XTH)=$(r/\lambda,\theta)$ where $(r,\theta)$ are polar coordinates of the location where $\phi$ is to be computed, and $\lambda$ is the incident wave length. Therefore, XRL is the radius vector or radius distance normalized by the incident wave length. XTH is the vectorial angle in degree.

b. WEDGEA = wedge angle in degree.

c. WAVEA = incident wave angle in degree.

d. IDX = an index (set to 0 in this subroutine).

Output arguments are as follows:

---

* References cited in the Appendix can be found in the References at the end of the main text.

a. F = $\phi$ in Equation 26 (wave response normalized by incident wave amplitude).

b. FABS = amplification factor, the absolute value of $\phi$ .

c. FPHA = phase difference, the phase of $\phi$ .

## Example and Test Run

3. To serve as an example as well as to ensure the subroutine is the correct one, the user should run the test program listed in Figure A1 and make sure the output is the same as that listed in Table A1.

```
      PROGRAM WEDGE1(INPUT,OUTPUT,TAPE5=INPUT,TAPE6=OUTPUT)                TEST1
      COMPLEX F                                                            TEST2
      PI=3.141592654                                                       TEST3
      WRITE(6,4)                                                           TEST4
    4 FORMAT(//3X,' WEDG-ANG WAV-ANG       LOCATION       WAVE RESPONSE TEST5
     1 ABS-VAL   PHASE'/3X,'  (DEG)    (DEG)      XRL  XTH(DEG)    (NORMTEST6
     2ALIZED)   AMP-FAC   (RAD)'/3X,35(' -')//)                           TEST7
      WEDGEA=90.                                                           TEST8
      WAVEA=135.                                                           TEST9
      XRL=2.0                                                              TEST10
      XTH=30.                                                              TEST11
      IDX=0                                                                TEST12
      CALL WEDGE(F,FABS,FPHA,XRL,XTH,WEDGEA,WAVEA,IDX)                     TEST13
      WRITE(6,40) WEDGEA,WAVEA,XRL,XTH,F,FABS,FPHA                         TEST14
   40 FORMAT(1X,8F9.2)                                                     TEST15
      STOP                                                                 TEST16
      END                                                                  TEST17
```

Figure A1. Computer program list 1

Table A1

Sample Output of the Test Program

| WEDG-ANG (DEG) | WAV-ANG (DEG) | LOCATION XRL | LOCATION XTH(DEG) | WAVE RESPONSE (NORMALIZED) | ABS-VAL AMP-FAC | PHASE (RAD) |
|---|---|---|---|---|---|---|
| 90.00 | 135.00 | 2.00 | 30.00 | -.41 | .68 | .79 | 2.11 |

4. The input arguments in the test programs are as follows:

   a. WEDGEA = 90; the wedge angle is 90 deg.

   b. WAVEA = 135; the incident wave angle is 135 deg.

   c. XRL = 2.0 and XTH = 30; the location of the wave response $\phi$ to be computed is at radial vector of two incident wave length distances and vectorial angle of 30 deg in polar coordinates.

   d. IDX = 0; the index IDX is set to 0.

This case is shown in Figure A2. The outputs are given in Table A1 which is self-explanatory.

5. If the location of the wave response $\phi$ to be computed is a very large distance, for example XRL greater than 18 or so, the output might print the message that the number of terms is insufficient for summation in computing $\phi$. In this situation, the user must replace the integer 200 in PARAMETER(NN=200), listed in Card WEDGE15 in the SUBROUTINE WEDGE list, by a larger integer to ensure the accuracy.

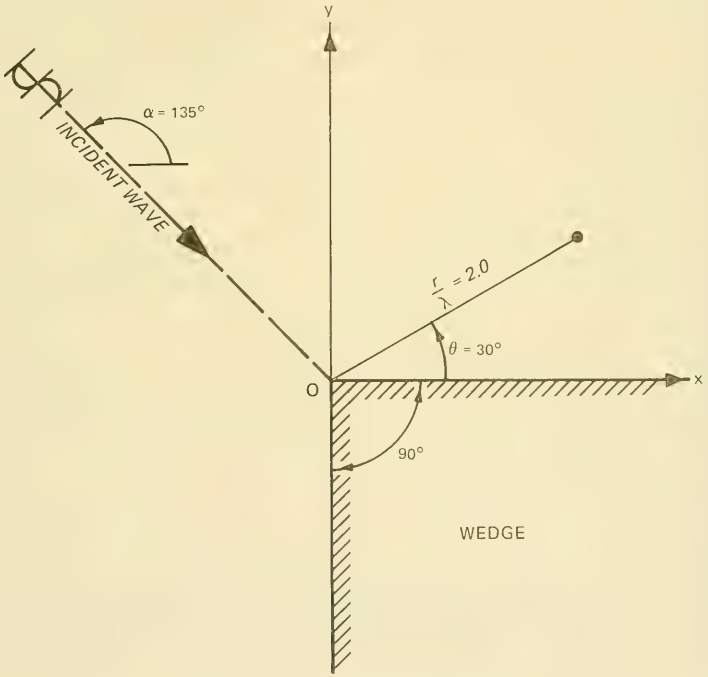


Figure A2. Example problem

## Subroutine WEDGE Listing

6. Subroutine WEDGE, which is listed in this section, calls subroutine BESJ which, in turn, calls subroutine JAIRY and function GAMLN. Subroutines BESJ and JAIRY and function GAMLN are borrowed from the NSWC Library of Mathematics Subroutines (Morris 1984). Including these borrowed subroutines here in the list is only for the purpose of allowing subroutine WEDGE to be self-contained and complete. The computer program of subroutine WEDGE is listed in Figure A3.

```
      SUBROUTINE WEDGE(F,FABS,FPHA,XRL,XTH,WEDGEA,WAVEA,IDX)            WEGDE1
C  * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *NOTICE1
C  *     THIS COMPUTER PROGRAM WAS WRITTEN BY H.S. CHEN OF CERC IN      *NOTICE2
C  *     1986 UNDER CORP' CIVIL WORK R&D PROGRAM.  NEITHER ANY OF       *NOTICE3
C  *     AGENCIES NOR ANY INDIVIDUAL ASSUMES ANY LEGAL LIABILITY OR     *NOTICE4
C  *     RESPONSIBILITY FOR THE ACCURACY OF THE PROGRAM.                *NOTICE5
C  * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *NOTICE6
C  - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - -WEDGE2
C    WAVE REFLECTION AND DIFFRACTION BY A VERTICAL WEDGE.                WEDGE3
C                                                                        WEDGE4
C       INPUT :     XRL = R/L, (RADIUS VECTOR)/(WAVE LENGTH)            WEDGE5
C                   XTH = VECTORIAL ANGLE IN DEGREE                     WEDGE6
C                WEDGEA = WEDGE ANGLE IN DEGREE                         WEDGE7
C                 WAVEA = INCIDENT WAVE ANGLE IN DEGREE                 WEDGE8
C      OUTPUT :       F = PHI, WAVE RESPONSE NORMALIZED BY INCIDENT     WEDGE9
C                         WAVE AMPLITUDE                               WEDGE10
C                  FABS = ABSOLUTE VALUE OF PHI, THE AMPLIFICATION     WEDGE11
C                         FACTOR                                       WEDGE12
C                  FPHA = PHASE OF PHI IN RADIAN.                      WEDGE13
C  - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - -WEDGE14
      PARAMETER(NN=200)                                                 WEDGE15
      DIMENSION BJ(NN),W(NN),XM(NN)                                     WEDGE16
      COMPLEX F,TM                                                      WEDGE17
      DATA TOLR/1.E-8/,ITER/8/                                          WEDGE18
      PI=3.141592654                                                    WEDGE19
      CPI=PI/180.                                                       WEDGE20
      XKR=XRL*2.0*PI                                                    WEDGE21
      TH=XTH*CPI                                                        WEDGE22
      WA=WAVEA*CPI                                                      WEDGE23
      XNU=(360.-WEDGEA)/180.                                            WEDGE24
      XM(1)=1.0                                                         WEDGE25
      IF(IDX.NE.0) GOTO 14                                              WEDGE26
      CALL BESJ(XKR,0.0,1,W,NZ)                                         WEDGE27
      BJ(1)=W(1)                                                        WEDGE28
      ICOUNT=0                                                          WEDGE29
      DO 10 N=1,NN                                                      WEDGE30
      N1=N+1                                                            WEDGE31
      ICOUNT=ICOUNT+1                                                   WEDGE32
      IF(ICOUNT.LE.ITER) GOTO 8                                         WEDGE33
      NNN=N                                                             WEDGE34
      GOTO 14                                                           WEDGE35
    8 XM(N1)=FLOAT(N)/XNU                                               WEDGE36
      M=INT(XM(N1))                                                     WEDGE37
      ALPHA=XM(N1)-M                                                    WEDGE38
      M1=M+1                                                            WEDGE39
      CALL BESJ(XKR,ALPHA,M1,W,NZ)                                      WEDGE40
      IF(N1.EQ.NN) WRITE(6,9) XKR,ALPHA,M1,W(M1),NZ                     WEDGE41
    9 FORMAT(/' **** NO. OF TERMS FOR SUMMATION IS INSUFFICIENT ****',  WEDGE42
     1 /' XKR,ALPHA,M1,W(M1),NZ =',2F10.4,I5,E15.6,I5/)                 WEDGE43
      BJ(N1)=W(M1)                                                      WEDGE44
```

Figure A3. Computer program list 2 (Sheet 1 of 25)

```
      IF(ABS(BJ(N1)).GT.TOLR) ICOUNT=0                                  WEDGE45
   10 CONTINUE                                                          WEDGE46
   14 CONTINUE                                                          WEDGE47
      F=BJ(1)/2.                                                        WEDGE48
      DO 20 N=1,NNN                                                     WEDGE49
      N1=N+1                                                            WEDGE50
      XMN=XM(N1)                                                        WEDGE51
      TM=(0.0,1.0)**XMN*BJ(N1)*COS(XMN*WA)*COS(XMN*TH)                  WEDGE52
      F=F+TM                                                            WEDGE53
   20 CONTINUE                                                          WEDGE54
      F=4./XNU*F                                                        WEDGE55
      FR=REAL(F)                                                        WEDGE56
      FI=AIMAG(F)                                                       WEDGE57
      FABS=SQRT(FR*FR+FI*FI)                                            WEDGE58
      IF(WA.LE.1.E-8) FABS=FABS/2.                                      WEDGE59
      IF(FABS.LT.TOLR) GOTO 30                                          WEDGE60
      FPHA=ATAN2(FI,FR)                                                 WEDGE61
      RETURN                                                            WEDGE62
   30 FPHA=0.0                                                          WEDGE63
      RETURN                                                            WEDGE64
      END                                                               WEDGE65
      SUBROUTINE BESJ(X,ALPHA,N,Y,NZ)                                   BESJ101
C  - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - -BESJ102
C     WRITTEN BY D.E. AMOS, S.L. DANIEL AND M.K. WESTON, JANUARY, 1975. BESJ103
C     REFERENCE SAND-75-0147                                            BESJ104
C                                                                       BESJ105
C     ABSTRACT                                                          BESJ106
C         BESJ COMPUTES AN N MEMBER SEQUENCE OF J BESSEL FUNCTIONS      BESJ107
C         J/SUB(ALPHA+K-1)/(X), K=1,...,N FOR NON-NEGATIVE ALPHA AND X. BESJ108
C         A COMBINATION OF THE POWER SERIES, THE ASYMPTOTIC EXPANSION   BESJ109
C         FOR X TO INFINITY AND THE UNIFORM ASYMPTOTIC EXPANSION FOR    BESJ110
C         NU TO INFINITY ARE APPLIED OVER SUBDIVISIONS OF THE (NU,X)    BESJ111
C         PLANE. FOR VALUES OF (NU,X) NOT COVERED BY ONE OF THESE       BESJ112
C         FORMULAE, THE ORDER IS INCREMENTED OR DECREMENTED BY INTEGER  BESJ113
C         VALUES INTO A REGION WHERE ONE OF THE FORMULAE APPLY. BACKWARDBESJ114
C         RECURSION IS APPLIED TO REDUCE ORDERS BY INTEGER VALUES EXCEPTBESJ115
C         WHERE THE ENTIRE SEQUENCE LIES IN THE OSCILLATORY REGION. IN  BESJ116
C         THIS CASE FORWARD RECURSION IS STABLE AND VALUES FROM THE     BESJ117
C         ASYMPTOTIC EXPANSION FOR X TO INFINITY START THE RECURSION    BESJ118
C         WHEN IT IS EFFICIENT TO DO SO. LEADING TERMS OF THE SERIES ANDBESJ119
C         UNIFORM EXPANSION ARE TESTED FOR UNDERFLOW. IF A SEQUENCE IS  BESJ120
C         REQUESTED AND THE LAST MEMBER WOULD UNDERFLOW, THE RESULT IS  BESJ121
C         SET TO ZERO AND THE NEXT LOWER ORDER TRIED, ETC., UNTIL A     BESJ122
C         MEMBER COMES ON SCALE OR ALL MEMBERS ARE SET TO ZERO. OVERFLOWBESJ123
C         CANNOT OCCUR. BESJ1 CALLS SUBROUTINE JAIRY AND FUNCTION GAMLN.BESJ124
C                                                                       BESJ125
C     DESCRIPTION OF ARGUMENTS                                          BESJ126
C                                                                       BESJ127
C         INPUT                                                         BESJ128
C           X      - X.GE.0                                           BESJ129
C           ALPHA  - ORDER OF FIRST MEMBER OF THE SEQUENCE, ALPHA.GE.0 BESJ130
C           N      - NUMBER OF MEMBERS IN THE SEQUENCE, N.GE.1        BESJ131
C         OUTPUT                                                        BESJ132
C           Y      - A VECTOR WHOSE FIRST N COMPONENTS CONTAIN        BESJ133
C                    VALUES FOR J/SUB(ALPHA+K-1)/(X), K=1,...,N       BESJ134
```

Figure A3. (Sheet 2 of 25)

```
C           NZ     - ERROR INDICATOR                                      BESJ135
C                    NZ=0       NORMAL RETURN - COMPUTATION COMPLETED     BESJ136
C                    NZ=-1      X IS LESS THAN 0.0                        BESJ137
C                    NZ=-2      ALPHA IS LESS THAN 0.0                    BESJ138
C                    NZ=-3      N IS LESS THAN 1                          BESJ139
C                    NZ.GT.0    LAST NZ COMPONENTS OF Y SET TO 0.0        BESJ140
C                               BECAUSE OF UNDERFLOW                      BESJ141
C                                                                         BESJ142
C     ERROR CONDITIONS                                                    BESJ143
C                                                                         BESJ144
C         IMPROPER INPUT ARGUMENTS - A FATAL ERROR                        BESJ145
C         UNDERFLOW  - A NON-FATAL ERROR (NZ.GT.0)                        BESJ146
C                                                                         BESJ147
C  - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - -  -BESJ148
C                                                                         BESJ149
      DOUBLE PRECISION DX,TRX,DTM,DFN                                     BESJ150
      DIMENSION Y(N)                                                      BESJ151
      DIMENSION C(11,10),ALFA(26,4),BETA(26,5)                            BESJ152
      DIMENSION C1(88),C2(22)                                             BESJ153
      DIMENSION A1(52),A2(52),B1(52),B2(52),B3(26)                        BESJ154
      DIMENSION GAMA(26),TEMP(3),KMAX(5),AR(8),BR(10),UPOL(10)            BESJ155
      DIMENSION FNULIM(2),PP(4)                                           BESJ156
      DIMENSION CR(10),DR(10)                                             BESJ157
C                                                                         BESJ158
      EQUIVALENCE (C(1,1),C1(1))                                          BESJ159
      EQUIVALENCE (C(1,9),C2(1))                                          BESJ160
      EQUIVALENCE (ALFA(1,1),A1(1))                                       BESJ161
      EQUIVALENCE (ALFA(1,3),A2(1))                                       BESJ162
      EQUIVALENCE (BETA(1,1),B1(1))                                       BESJ163
      EQUIVALENCE (BETA(1,3),B2(1))                                       BESJ164
      EQUIVALENCE (BETA(1,5),B3(1))                                       BESJ165
C                                                                         BESJ166
      DATA ELIM1,ELIM2,TOL /   667.    ,    644.    ,    1.E-15        /BESJ167
C                                                                         BESJ168
      DATA PP(1)/8.729091539355E+00/,  PP(2)/2.6569393226503E-01/,        BESJ169
     1     PP(3)/1.2457857686559E-01/,  PP(4)/7.7013374743039E-04/       BESJ170
C                                                                         BESJ171
C     TOLS=LN(1.E-3)                                                      BESJ172
      DATA TOLS            /-6.9077552789821E+00/                         BESJ173
C                                                                         BESJ174
      DATA UPOL(1),CON1,CON2,CON3,CON548         / 1.0000000000000E+00, BESJ175
     1 6.6666666666667E-01, 3.3333333333333E-01, 1.4142135623731E+00,     BESJ176
     2 1.0416666666667E-01/                                               BESJ177
C                                                                         BESJ178
      DATA RTWO,PDF,RTTP,PIDT                    / 1.3483997249265E+00, BESJ179
     1 7.8539816339745E-01, 7.9788456080286E-01, 1.5707963267949E+00/     BESJ180
C                                                                         BESJ181
      DATA FNULIM(1)/100./,  FNULIM(2)/60./                               BESJ182
C                                                                         BESJ183
C     CE=-ALOG(TOL) , TCE=-0.75*ALOG(TOL)                                 BESJ184
      DATA CE , TCE        / 3.4538776394911E+01, 2.5904082296183E+01/    BESJ185
C                                                                         BESJ186
      DATA INLIM           /         150         /                        BESJ187
C                                                                         BESJ188
      DATA AR(1)/8.3550347222222E-02/,  AR(2)/1.2822657455633E-01/,       BESJ189A
```

Figure A3. (Sheet 3 of 25)

```
     1     AR(3)/2.9184902646414E-01/,  AR(4)/8.8162726744376E-01/,     BESJ189B
     2     AR(5)/3.3214082818628E+00/,  AR(6)/1.4995762986863E+01/,     BESJ189C
     3     AR(7)/7.8923013011586E+01/,  AR(8)/4.7445153886826E+02/      BESJ189D
C                                                                       BESJ190
      DATA BR(1) /-1.4583333333333E-01/, BR(2) /-9.8741319444444E-02/,  BSEJ191A
     1     BR(3) /-1.4331205391590E-01/, BR(4) /-3.1722720267841E-01/,  BSEJ191B
     2     BR(5) /-9.4242914795712E-01/, BR(6) /-3.5112030408264E+00/,  BSEJ191C
     3     BR(7) /-1.5727263620368E+01/, BR(8) /-8.2281439097186E+01/,  BSEJ191D
     4     BR(9) /-4.9235537052367E+02/, BR(10)/-3.3162185685480E+03/   BSEJ191E
C                                                                       BESJ192
      DATA C1(1) /-2.0833333333333E-01/, C1(2) / 1.2500000000000E-01/,  BESJ193A
     1     C1(3) / 0.0/, C1(4) / 0.0/, C1(5) / 0.0/, C1(6) / 0.0/,      BESJ193B
     2     C1(7) / 0.0/, C1(8) / 0.0/, C1(9) / 0.0/, C1(10)/ 0.0/,      BESJ193C
     3     C1(11)/ 0.0/, C1(12)/ 3.3420138888889E-01/,                  BESJ193D
     4     C1(13)/-4.0104166666667E-01/, C1(14)/ 7.0312500000000E-02/,  BESJ193E
     5     C1(15)/ 0.0/, C1(16)/ 0.0/, C1(17)/ 0.0/, C1(18)/ 0.0/,      BESJ193F
     6     C1(19)/ 0.0/, C1(20)/ 0.0/, C1(21)/ 0.0/, C1(22)/ 0.0/,      BESJ193G
     7     C1(23)/-1.0258125964506E+00/, C1(24)/ 1.8464626736111E+00/,  BESJ193H
     8     C1(25)/-8.9121093750000E-01/, C1(26)/ 7.3242187500000E-02/,  BESJ193I
     9     C1(27)/ 0.0/, C1(28)/ 0.0/, C1(29)/ 0.0/, C1(30)/ 0.0/,      BESJ193J
     1     C1(31)/ 0.0/, C1(32)/ 0.0/, C1(33)/ 0.0/,                    BESJ193K
     2     C1(34)/ 4.6695844234262E+00/, C1(35)/-1.1207002616223E+01/,  BESJ193L
     3     C1(36)/ 8.7891235351562E+00/, C1(37)/-2.3640869140625E+00/,  BESJ193M
     4     C1(38)/ 1.1215209960938E-01/, C1(39)/ 0.0/, C1(40)/ 0.0/,    BESJ193N
     5     C1(41)/ 0.0/, C1(42)/ 0.0/, C1(43)/ 0.0/, C1(44)/ 0.0/,      BESJ193O
     6     C1(45)/-2.8212072558200E+01/, C1(46)/ 8.4636217674601E+01/,  BESJ193P
     7     C1(47)/-9.1818241543240E+01/, C1(48)/ 4.2534998745388E+01/,  BESJ931Q
     8     C1(49)/-7.3687943594796E+00/, C1(50)/ 2.2710800170898E-01/,  BESJ193R
     9     C1(51)/ 0.0/, C1(52)/ 0.0/, C1(53)/ 0.0/, C1(54)/ 0.0/       BESJ193S
C                                                                       BESJ194
      DATA C1(55)/ 0.0/, C1(56)/ 2.125701300392 2E+02/,                 BESJ195A
     1     C1(57)/-7.6525246814118E+02/, C1(58)/ 1.0599904525280E+03/,  BESJ195B
     2     C1(59)/-6.9957962737613E+02/, C1(60)/ 2.1819051174421E+02/,  BESJ195C
     3     C1(61)/-2.6491430486952E+01/, C1(62)/ 5.7250142097473E-01/,  BESJ195D
     4     C1(63)/ 0.0/, C1(64)/ 0.0/, C1(65)/ 0.0/, C1(66)/ 0.0/,      BESJ195E
     5     C1(67)/-1.9194576623184E+03/, C1(68)/ 8.0617221817373E+03/,  BESJ195F
     6     C1(69)/-1.3586550006434E+04/, C1(70)/ 1.1655393336864E+04/,  BESJ195G
     7     C1(71)/-5.3056469786134E+03/, C1(72)/ 1.2009029132164E+03/,  BESJ195H
     8     C1(73)/-1.0809091978840E+02/, C1(74)/ 1.7277275025845E+00/,  BESJ195I
     9     C1(75)/ 0.0/, C1(76)/ 0.0/, C1(77)/ 0.0/,                    BESJ195J
     1     C1(78)/ 2.0204291330966E+04/, C1(79)/-9.6980598388638E+04/,  BESJ195K
     2     C1(80)/ 1.9254700123253E+05/, C1(81)/-2.0340017728042E+05/,  BESJ195L
     3     C1(82)/ 1.2220046498302E+05/, C1(83)/-4.1192654968898E+04/,  BESJ195M
     4     C1(84)/ 7.1095143024894E+03/, C1(85)/-4.9391530477309E+02/,  BESJ195N
     5     C1(86)/ 6.0740420012735E+00/, C1(87)/ 0.0/, C1(88)/ 0.0/     BESJ195O
C                                                                       BESJ196
      DATA C2(1) /-2.4291918790055E+05/, C2(2) / 1.3117636146630E+06/,  BESJ197A
     1     C2(3) /-2.9980159185381E+06/, C2(4) / 3.7632712976564E+06/,  BESJ197B
     2     C2(5) /-2.8135632265865E+06/, C2(6) / 1.2683652733216E+06/,  BESJ197C
     3     C2(7) /-3.3164517248456E+05/, C2(8) / 4.5218768981363E+04/,  BESJ197D
     4     C2(9) /-2.4998304818112E+03/, C2(10)/ 2.4380529699556E+01/,  BESJ197E
     5     C2(11)/ 0.0/, C2(12)/ 3.2844698530720E+06/,                  BESJ197F
     6     C2(13)/-1.9706819118432E+07/, C2(14)/ 5.0952602492665E+07/,  BESJ197G
     7     C2(15)/-7.4105148211533E+07/, C2(16)/ 6.6344512274729E+07/,  BESJ197H
     8     C2(17)/-3.7567176660763E+07/, C2(18)/ 1.3288767166422E+07/,  BESJ197I
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Figure A3. (Sheet 4 of 25)

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     9     C2(19)/-2.7856181280864E+06/, C2(20)/ 3.0818640461266E+05/,  BESJ197J
     1     C2(21)/-1.3886089753717E+04/, C2(22)/ 1.1001714026925E+02/   BESJ197K
C                                                                       BESJ198
      DATA A1(1) /-4.4444444444444E-03/, A1(2) /-9.2207792207792E-04/,  BESJ199A
     1     A1(3) /-8.8489288489288E-05/, A1(4) / 1.6592768783245E-04/,  BESJ199B
     2     A1(5) / 2.4669137274179E-04/, A1(6) / 2.6599558934626E-04/,  BESJ199C
     3     A1(7) / 2.6182429706150E-04/, A1(8) / 2.4873043734466E-04/,  BESJ199D
     4     A1(9) / 2.3272104008323E-04/, A1(10)/ 2.1636248571236E-04/,  BESJ199E
     5     A1(11)/ 2.0073885876275E-04/, A1(12)/ 1.8626763663754E-04/,  BESJ199F
     6     A1(13)/ 1.7306077591788E-04/, A1(14)/ 1.6109170592902E-04/,  BESJ199G
     7     A1(15)/ 1.5027477416091E-04/, A1(16)/ 1.4050349739127E-04/,  BESJ199H
     8     A1(17)/ 1.3166881654592E-04/, A1(18)/ 1.2366744559825E-04/,  BESJ199I
     9     A1(19)/ 1.1640527147474E-04/, A1(20)/ 1.0979829837271E-04/,  BESJ199J
     1     A1(21)/ 1.0377241042299E-04/, A1(22)/ 9.8262607836936E-05/,  BESJ199K
     2     A1(23)/ 9.3212051724950E-05/, A1(24)/ 8.8571085247871E-05/,  BESJ199L
     3     A1(25)/ 8.4296310571570E-05/, A1(26)/ 8.0349754840779E-05/,  BESJ199M
     4     A1(27)/ 6.9373554135459E-04/, A1(28)/ 2.3224174518292E-04/,  BESJ199N
     5     A1(29)/-1.4198627355669E-05/, A1(30)/-1.1644493167205E-04/,  BESJ199O
     6     A1(31)/-1.5080355805305E-04/, A1(32)/-1.5512192491810E-04/,  BESJ199P
     7     A1(33)/-1.4680975664647E-04/, A1(34)/-1.3381550386749E-04/,  BESJ199Q
     8     A1(35)/-1.1974497568425E-04/, A1(36)/-1.0618431920797E-04/,  BESJ199R
     9     A1(37)/-9.3769954989119E-05/, A1(38)/-8.2692304558819E-05/   BESJ199S
C                                                                       BESJ200
      DATA A1(39)/-7.2937434815522E-05/, A1(40)/-6.4404235772102E-05/,  BESJ201A
     1     A1(41)/-5.6961156600937E-05/, A1(42)/-5.0473104430356E-05/,  BESJ201B
     2     A1(43)/-4.4813486800888E-05/, A1(44)/-3.9868872771760E-05/,  BESJ201C
     3     A1(45)/-3.5540053297204E-05/, A1(46)/-3.1741425660902E-05/,  BESJ201D
     4     A1(47)/-2.8399679390418E-05/, A1(48)/-2.5452272063487E-05/,  BESJ201E
     5     A1(49)/-2.2845929716472E-05/, A1(50)/-2.0535275310648E-05/,  BESJ201F
     6     A1(51)/-1.8481621762767E-05/, A1(52)/-1.6651933002139E-05/   BESJ201G
C                                                                       BESJ202
      DATA A2(1) /-3.5421197145774E-04/, A2(2) /-1.5616126394516E-04/,  BESJ203A
     1     A2(3) / 3.0446550359494E-05/, A2(4) / 1.3019865577324E-04/,  BESJ203B
     2     A2(5) / 1.6747110669971E-04/, A2(6) / 1.7022258768359E-04/,  BESJ203C
     3     A2(7) / 1.5650142760860E-04/, A2(8) / 1.3633917097744E-04/,  BESJ203D
     4     A2(9) / 1.1488669202982E-04/, A2(10)/ 9.4586909303469E-05/,  BESJ203E
     5     A2(11)/ 7.6449841925090E-05/, A2(12)/ 6.0757033496520E-05/,  BESJ203F
     6     A2(13)/ 4.7439429929051E-05/, A2(14)/ 3.6275751200534E-05/,  BESJ203G
     7     A2(15)/ 2.6993971497922E-05/, A2(16)/ 1.9321093824794E-05/,  BESJ203H
     8     A2(17)/ 1.3005667479396E-05/, A2(18)/ 7.8626086674450E-06/,  BESJ203I
     9     A2(19)/ 3.5925748581935E-06/, A2(20)/ 1.4404004981425E-07/,  BESJ203J
     1     A2(21)/-2.6539676969794E-06/, A2(22)/-4.9134686709849E-06/,  BESJ203K
     2     A2(23)/-6.7273929609125E-06/, A2(24)/-8.1726937967866E-06/,  BESJ203L
     3     A2(25)/-9.3130471509356E-06/, A2(26)/-1.0201141879802E-05/,  BESJ203M
     4     A2(27)/ 3.7819419920177E-04/, A2(28)/ 2.0247195276182E-04/,  BESJ203N
     5     A2(29)/-6.3793850631886E-05/, A2(30)/-2.3859823060301E-04/,  BESJ203O
     6     A2(31)/-3.1091625602736E-04/, A2(32)/-3.1368011524758E-04/,  BESJ203P
     7     A2(33)/-2.7895027379132E-04/, A2(34)/-2.2856408261914E-04/,  BESJ203Q
     8     A2(35)/-1.7524528034085E-04/, A2(36)/-1.2554406306069E-04/,  BESJ203R
     9     A2(37)/-8.2298287282021E-05/, A2(38)/-4.6286073058812E-05/   BESJ203S
C                                                                       BESJ204
      DATA A2(39)/-1.7233430236696E-05/, A2(40)/ 5.6069048230460E-06/,  BESJ205A
     1     A2(41)/ 2.3139544314829E-05/, A2(42)/ 3.6264274585679E-05/,  BESJ205B
     2     A2(43)/ 4.5800612449019E-05/, A2(44)/ 5.2459529495911E-05/,  BESJ205C
     3     A2(45)/ 5.6839620854582E-05/, A2(46)/ 5.9434982039310E-05/,  BESJ205D
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Figure A3. (Sheet 5 of 25)

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     4     A2(47)/ 6.0647852757842E-05/, A2(48)/ 6.0802390778844E-05/,  BESJ205E
     5     A2(49)/ 6.0157789453946E-05/, A2(50)/ 5.8919965734470E-05/,  BESJ205F
     6     A2(51)/ 5.7251582377759E-05/, A2(52)/ 5.5280437558585E-05/   BESJ205G
C                                                                       BESJ206
      DATA B1(1) / 1.7998872141355E-02/, B1(2) / 5.5996491106439E-03/,  BESJ207A
     1     B1(3) / 2.8850140223113E-03/, B1(4) / 1.8009660676105E-03/,  BESJ207B
     2     B1(5) / 1.2475311058920E-03/, B1(6) / 9.2287887657294E-04/,  BESJ207C
     3     B1(7) / 7.1443042172729E-04/, B1(8) / 5.7178728178970E-04/,  BESJ207D
     4     B1(9) / 4.6943100760648E-04/, B1(10)/ 3.9323283546292E-04/,  BESJ207E
     5     B1(11)/ 3.3481888931830E-04/, B1(12)/ 2.8895214849575E-04/,  BESJ207F
     6     B1(13)/ 2.5221161554957E-04/, B1(14)/ 2.2228058079888E-04/,  BESJ207G
     7     B1(15)/ 1.9754183803306E-04/, B1(16)/ 1.7683685501972E-04/,  BESJ207H
     8     B1(17)/ 1.5931689966182E-04/, B1(18)/ 1.4434793019733E-04/,  BESJ207I
     9     B1(19)/ 1.3144806811996E-04/, B1(20)/ 1.2024544494930E-04/,  BESJ207J
     1     B1(21)/ 1.1044914450460E-04/, B1(22)/ 1.0182877074057E-04/,  BESJ207K
     2     B1(23)/ 9.4199822420424E-05/, B1(24)/ 8.7413054575383E-05/,  BESJ207L
     3     B1(25)/ 8.1346626216280E-05/, B1(26)/ 7.5900226964622E-05/,  BESJ207M
     4     B1(27)/-1.4928295321343E-03/, B1(28)/-8.7820470954639E-04/,  BESJ207N
     5     B1(29)/-5.0291654957204E-04/, B1(30)/-2.9482213851275E-04/,  BESJ207O
     6     B1(31)/-1.7546399697078E-04/, B1(32)/-1.0400855046082E-04/,  BESJ207P
     7     B1(33)/-5.9614195304646E-05/, B1(34)/-3.1203892907610E-05/,  BESJ207Q
     8     B1(35)/-1.2608973598023E-05/, B1(36)/-2.4289260857573E-07/,  BESJ207R
     9     B1(37)/ 8.0599616541427E-06/, B1(38)/ 1.3650700926215E-05/   BESJ207S
C                                                                       BESJ208
      DATA B1(39)/ 1.7396412547293E-05/, B1(40)/ 1.9867297884213E-05/,  BESJ209A
     1     B1(41)/ 2.1446326379082E-05/, B1(42)/ 2.2395465923246E-05/,  BESJ209B
     2     B1(43)/ 2.2896778381471E-05/, B1(44)/ 2.3078538981118E-05/,  BESJ209C
     3     B1(45)/ 2.3032197608091E-05/, B1(46)/ 2.2823607372035E-05/,  BESJ209D
     4     B1(47)/ 2.2500588110529E-05/, B1(48)/ 2.2098101536199E-05/,  BESJ209E
     5     B1(49)/ 2.1641842744810E-05/, B1(50)/ 2.1150764925622E-05/,  BESJ209F
     6     B1(51)/ 2.0638874978217E-05/, B1(52)/ 2.0116524199708E-05/   BESJ209G
C                                                                       BESJ210
      DATA B2(1) / 5.5221307672129E-04/, B2(2) / 4.4793258155238E-04/,  BESJ211A
     1     B2(3) / 2.7952065399202E-04/, B2(4) / 1.5246815619845E-04/,  BESJ211B
     2     B2(5) / 6.9327110565704E-05/, B2(6) / 1.7625868306999E-05/,  BESJ211C
     3     B2(7) /-1.3574499634327E-05/, B2(8) /-3.1797241335043E-05/,  BESJ211D
     4     B2(9) /-4.1886186169669E-05/, B2(10)/-4.6900488937914E-05/,  BESJ211E
     5     B2(11)/-4.8766544741379E-05/, B2(12)/-4.8701003118674E-05/,  BESJ211F
     6     B2(13)/-4.7475562089009E-05/, B2(14)/-4.5581305813863E-05/,  BESJ211G
     7     B2(15)/-4.3330964451127E-05/, B2(16)/-4.0923019315775E-05/,  BESJ211H
     8     B2(17)/-3.8482263860322E-05/, B2(18)/-3.6085716753541E-05/,  BESJ211I
     9     B2(19)/-3.3779330612337E-05/, B2(20)/-3.1588856077211E-05/,  BESJ211J
     1     B2(21)/-2.9526956175081E-05/, B2(22)/-2.7597891482834E-05/,  BESJ211K
     2     B2(23)/-2.5800617466688E-05/, B2(24)/-2.4130835676128E-05/,  BESJ211L
     3     B2(25)/-2.2582350951835E-05/, B2(26)/-2.1147965676891E-05/,  BESJ211M
     4     B2(27)/-4.7461779655996E-04/, B2(28)/-4.7786456714732E-04/,  BESJ211N
     5     B2(29)/-3.2039022806704E-04/, B2(30)/-1.6110501611996E-04/,  BESJ211O
     6     B2(31)/-4.2577810128544E-05/, B2(32)/ 3.4457129429497E-05/,  BESJ211P
     7     B2(33)/ 7.9709268407568E-05/, B2(34)/ 1.0313823670827E-04/,  BESJ211Q
     8     B2(35)/ 1.1246677526220E-04/, B2(36)/ 1.1310364210848E-04/,  BESJ211R
     9     B2(37)/ 1.0865163484877E-04/, B2(38)/ 1.0143795159766E-04/   BESJ211S
C                                                                       BESJ212
      DATA B2(39)/ 9.2929839659336E-05/, B2(40)/ 8.4029313301609E-05/,  BESJ213A
     1     B2(41)/ 7.5272799134913E-05/, B2(42)/ 6.6963252197573E-05/,  BESJ213B
     2     B2(43)/ 5.9256454732320E-05/, B2(44)/ 5.2216930882698E-05/,  BESJ213C
```

Figure A3. (Sheet 6 of 25)

```
     3     B2(45)/ 4.5853948516536E-05/, B2(46)/ 4.0144551389149E-05/,  BESJ213D
     4     B2(47)/ 3.5048173003133E-05/, B2(48)/ 3.0515799503435E-05/,  BESJ213E
     5     B2(49)/ 2.6495611995052E-05/, B2(50)/ 2.2936363369100E-05/,  BESJ213F
     6     B2(51)/ 1.9789305666402E-05/, B2(52)/ 1.7009198463641E-05/   BESJ213G
C                                                                       BESJ214
      DATA B3(1) / 7.3646581057258E-04/, B3(2) / 8.7279080514619E-04/,  BESJ215A
     1     B3(3) / 6.2261486257314E-04/, B3(4) / 2.8599815419430E-04/,  BESJ215B
     2     B3(5) / 3.8473767287937E-06/, B3(6) /-1.8790600363697E-04/,  BESJ215C
     3     B3(7) /-2.9760364659456E-04/, B3(8) /-3.4599812683267E-04/,  BESJ215D
     4     B3(9) /-3.5338247091604E-04/, B3(10)/-3.3571563577505E-04/,  BESJ215E
     5     B3(11)/-3.0432112478904E-04/, B3(12)/-2.6672272304761E-04/,  BESJ215F
     6     B3(13)/-2.2765421412282E-04/, B3(14)/-1.8992261185456E-04/,  BESJ215G
     7     B3(15)/-1.5505891859909E-04/, B3(16)/-1.2377824076187E-04/,  BESJ215H
     8     B3(17)/-9.6292614771764E-05/, B3(18)/-7.2517832771442E-05/,  BESJ215I
     9     B3(19)/-5.2207002889563E-05/, B3(20)/-3.5034775051190E-05/,  BESJ215J
     1     B3(21)/-2.0648976103555E-05/, B3(22)/-8.7010609684977E-06/,  BESJ215K
     2     B3(23)/ 1.1369868667510E-06/, B3(24)/ 9.1642647412278E-06/,  BESJ215L
     3     B3(25)/ 1.5647778542887E-05/, B3(26)/ 2.0822362948247E-05/   BESJ215M
C                                                                       BESJ216
      DATA GAMA(1) /6.2996052494744E-01/,                               BESJ217A
     1  GAMA(2) /2.5198420997898E-01/, GAMA(3) /1.5479030041566E-01/,  BESJ217B
     2  GAMA(4) /1.1071306241616E-01/, GAMA(5) /8.5730939552740E-02/,  BESJ217C
     3  GAMA(6) /6.9716131695868E-02/, GAMA(7) /5.8608567189371E-02/,  BESJ217D
     4  GAMA(8) /5.0469887353631E-02/, GAMA(9) /4.4260058068916E-02/,  BESJ217E
     5  GAMA(10)/3.9372066154351E-02/, GAMA(11)/3.5428319592446E-02/,  BESJ217F
     6  GAMA(12)/3.2181885750210E-02/, GAMA(13)/2.9464624079116E-02/,  BESJ217G
     7  GAMA(14)/2.7158167711293E-02/, GAMA(15)/2.5176827297386E-02/,  BESJ217H
     8  GAMA(16)/2.3457075530608E-02/, GAMA(17)/2.1950839013491E-02/,  BESJ217I
     9  GAMA(18)/2.0621082823565E-02/, GAMA(19)/1.9438824089788E-02/,  BESJ217J
     1  GAMA(20)/1.8381063380068E-02/, GAMA(21)/1.7429321323196E-02/,  BESJ217K
     2  GAMA(22)/1.6568583778661E-02/, GAMA(23)/1.5786528598792E-02/,  BESJ217L
     3  GAMA(24)/1.5072950149410E-02/, GAMA(25)/1.4419325083996E-02/,  BESJ217M
     4  GAMA(26)/1.3818480573534E-02/                                  BESJ217N
C     -------------------                                             BESJ218
C                                                                       BESJ219
C     TEST INPUT ARGUMENTS                                            BESJ220
C                                                                       BESJ221
      NZ=0                                                            BESJ222
      KT=1                                                            BESJ223
      IF(N-1) 92,108,109                                              BESJ224
  108 KT=2                                                            BESJ225
  109 NN=N                                                            BESJ226
      IF(X) 93,110,120                                                BESJ227
  110 IF(ALPHA) 91,114,116                                            BESJ228
  114 Y(1)=1.                                                         BESJ229
      IF(N.EQ.1) RETURN                                               BESJ230
      I1=2                                                            BESJ231
      GO TO 118                                                       BESJ232
  116 I1=1                                                            BESJ233
  118 DO 119 I=I1,N                                                   BESJ234
  119 Y(I)=0.                                                         BESJ235
      RETURN                                                          BESJ236
  120 CONTINUE                                                        BESJ237
      IF(ALPHA.LT.0.) GO TO 91                                        BESJ238
C                                                                     BESJ239
```

Figure A3. (Sheet 7 of 25)

```
      DFN=DBLE(FLOAT(N))+DBLE(ALPHA)-1.D+0                              BESJ240
      FNU=DFN                                                           BESJ241
      XO2=X*.5                                                          BESJ242
      SXO2=XO2*XO2                                                      BESJ243
C                                                                       BESJ244
C     DECISION TREE FOR REGION WHERE SERIES, ASYMPTOTIC EXPANSION FOR X BESJ245
C     TO INFINITY AND ASYMPTOTIC EXPANSION FOR NU TO INFINITY ARE       BESJ246
C     APPLIED.                                                          BESJ247
C                                                                       BESJ248
      IF(SXO2.LE.(FNU+1.)) GO TO 850                                    BESJ249
      TA=AMAX1(20.,FNU)                                                 BESJ250
      IF(X.GT.TA) GO TO 880                                             BESJ251
      IF(X.GT.12.) GO TO 860                                            BESJ252
      XO2L=ALOG(XO2)                                                    BESJ253
      NS=SXO2-FNU                                                       BESJ254
      GO TO 852                                                         BESJ255
  850 FN=FNU                                                            BESJ256
      FNP1=FN+1.                                                        BESJ257
      XO2L=ALOG(XO2)                                                    BESJ258
      IS=KT                                                             BESJ259
      IF(X.LE.0.5) GO TO 134                                            BESJ260
      NS=0                                                              BESJ261
  852 DFN=DFN+DBLE(FLOAT(NS))                                           BESJ262
      FN=DFN                                                            BESJ263
      FNP1=FN+1.                                                        BESJ264
      IS=KT                                                             BESJ265
      IF(N-1+NS.GT.0) IS=3                                              BESJ266
      GO TO 134                                                         BESJ267
  860 NS=AMAX1(36.-FNU,0.)                                              BESJ268
      DFN=DFN+DBLE(FLOAT(NS))                                           BESJ269
      FN=DFN                                                            BESJ270
      IS=KT                                                             BESJ271
      IF(N-1+NS.GT.0) IS=3                                              BESJ272
      GO TO 130                                                         BESJ273
  880 CONTINUE                                                          BESJ274
      RTX=SQRT(X)                                                       BESJ275
      TAU=RTWO*RTX                                                      BESJ276
      TA=TAU+FNULIM(KT)                                                 BESJ277
      IF(FNU.LE.TA) GO TO 500                                           BESJ278
  129 FN=FNU                                                            BESJ279
      IS=KT                                                             BESJ280
C                                                                       BESJ281
C     UNIFORM ASYMPTOTIC EXPANSION FOR NU TO INFINITY                   BESJ282
C                                                                       BESJ283
  130 CONTINUE                                                          BESJ284
      XX=X/FN                                                           BESJ285
      W2=1.-XX*XX                                                       BESJ286
      ABW2=ABS(W2)                                                      BESJ287
      RA=SQRT(ABW2)                                                     BESJ288
      IF(ABW2.GT.0.2775) GO TO 200                                      BESJ289
C                                                                       BESJ290
C     CASES NEAR X=FN, ABS(1.-(X/FN)**2).LE.0.2775                      BESJ291
C     COEFFICIENTS OF ASYMPTOTIC EXPANSION BY SERIES                    BESJ292
C                                                                       BESJ293
C                                                                       BESJ294
```

Figure A3. (Sheet 8 of 25)

```
C     ZETA AND TRUNCATION FOR A(ZETA) AND B(ZETA) SERIES                BESJ295
C                                                                       BESJ296
C     KMAX IS TRUNCATION INDEX FOR A(ZETA) AND B(ZETA) SERIES=MAX(2,SA) BESJ297
C                                                                       BESJ298
      SA=0.                                                             BESJ299
      IF(ABW2.EQ.0.) GO TO 21                                           BESJ300
      SA=TOLS/ALOG(ABW2)                                                BESJ301
   21 SB=SA                                                             BESJ302
      DO 22 I=1,5                                                       BESJ303
      KMAX(I)=AMAX1(SA,2.)                                              BESJ304
      SA=SA+SB                                                          BESJ305
   22 CONTINUE                                                          BESJ306
      KB=KMAX(5)                                                        BESJ307
      KLAST=KB-1                                                        BESJ308
      SA=GAMA(KB)                                                       BESJ309
      DO 24 K=1,KLAST                                                   BESJ310
      KB=KB-1                                                           BESJ311
      SA=SA*W2+GAMA(KB)                                                 BESJ312
   24 CONTINUE                                                          BESJ313
      Z=W2*SA                                                           BESJ314
      AZ=ABS(Z)                                                         BESJ315
      RTZ=SQRT(AZ)                                                      BESJ316
      FN13=FN**CON2                                                     BESJ317
      RTARY=RTZ*FN13                                                    BESJ318
      ARY=-RTARY*RTARY                                                  BESJ319
      AZ32=AZ*RTZ*CON1                                                  BESJ320
      ACZ=FN*AZ32                                                       BESJ321
      IF(Z.LE.0.) GO TO 27                                              BESJ322
C                                                                       BESJ323
C     TEST FOR UNDERFLOW, 1.E-280=EXP(-644.), ONE WORD LENGTH           BESJ324
C     UP FROM UNDERFLOW LIMIT OF CDC 6600                               BESJ325
C                                                                       BESJ326
      IF(ACZ.GT.ELIM2) GO TO 180                                        BESJ327
      ARY=-ARY                                                          BESJ328
   27 PHI=SQRT(SQRT(SA+SA+SA+SA))                                       BESJ329
C                                                                       BESJ330
C     B(ZETA) FOR S=0                                                   BESJ331
C                                                                       BESJ332
      KB=KMAX(5)                                                        BESJ333
      KLAST=KB-1                                                        BESJ334
      SB=BETA(KB,1)                                                     BESJ335
      DO 23 K=1,KLAST                                                   BESJ336
      KB=KB-1                                                           BESJ337
      SB=SB*W2+BETA(KB,1)                                               BESJ338
   23 CONTINUE                                                          BESJ339
      KSP1=1                                                            BESJ340
      FN2=FN*FN                                                         BESJ341
      RFN2=1./FN2                                                       BESJ342
      RDEN=1.                                                           BESJ343
      ASUM=1.                                                           BESJ344
      RELB=TOL*ABS(SB)                                                  BESJ345
      BSUM=SB                                                           BESJ346
      DO 25 KS=1,4                                                      BESJ347
      KSP1=KSP1+1                                                       BESJ348
      RDEN=RDEN*RFN2                                                    BESJ349
```

Figure A3. (Sheet 9 of 25)

```
C                                                                       BESJ350
C     A(ZETA) AND B(ZETA) FOR S=1,2,3,4                                 BESJ351
C                                                                       BESJ352
      KB=KMAX(5-KS)                                                     BESJ353
      KLAST=KB-1                                                        BESJ354
      SA=ALFA(KB,KS)                                                    BESJ355
      SB=BETA(KB,KSP1)                                                  BESJ356
      DO 26 K=1,KLAST                                                   BESJ357
      KB=KB-1                                                           BESJ358
      SA=SA*W2+ALFA(KB,KS)                                              BESJ359
      SB=SB*W2+BETA(KB,KSP1)                                            BESJ360
   26 CONTINUE                                                          BESJ361
      TA=SA*RDEN                                                        BESJ362
      TB=SB*RDEN                                                        BESJ363
      ASUM=ASUM+TA                                                      BESJ364
      BSUM=BSUM+TB                                                      BESJ365
      IF(ABS(TA).LE.TOL.AND.ABS(TB).LE.RELB) GO TO 152                  BESJ366
   25 CONTINUE                                                          BESJ367
  152 CONTINUE                                                          BESJ368
      BSUM=BSUM/(FN*FN13)                                               BESJ369
      GO TO 400                                                         BESJ370
C                                                                       BESJ371
  200 CONTINUE                                                          BESJ372
      TAU=1./RA                                                         BESJ373
      T2=1./W2                                                          BESJ374
      IF(W2.GE.0.) GO TO 30                                             BESJ375
C                                                                       BESJ376
C     CASES FOR (X/FN).GT.SQRT(1.2775)                                  BESJ377
C                                                                       BESJ378
      AZ32=ABS(RA-ATAN(RA))                                             BESJ379
      ACZ=AZ32*FN                                                       BESJ380
      CZ=-ACZ                                                           BESJ381
      Z32=1.5*AZ32                                                      BESJ382
      RTZ=Z32**CON2                                                     BESJ383
      FN13=FN**CON2                                                     BESJ384
      RTARY=RTZ*FN13                                                    BESJ385
      ARY=-RTARY*RTARY                                                  BESJ386
      GO TO 150                                                         BESJ387
   30 CONTINUE                                                          BESJ388
C                                                                       BESJ389
C     CASES FOR (X/FN).LT.SQRT(0.7225)                                  BESJ390
C                                                                       BESJ391
      AZ32=ABS(ALOG((1.+RA)/XX) -RA)                                    BESJ392
C                                                                       BESJ393
C     TEST FOR UNDERFLOW, 1.E-280 = EXP(-644.), ONE WORD LENGTH         BESJ394
C     UP FROM UNDERFLOW LIMIT OF CDC 6600                               BESJ395
C                                                                       BESJ396
      ACZ=AZ32*FN                                                       BESJ397
      CZ=ACZ                                                            BESJ398
      IF(ACZ.GT.ELIM2) GO TO 180                                        BESJ399
      Z32=1.5*AZ32                                                      BESJ400
      RTZ=Z32**CON2                                                     BESJ401
      FN13=FN**CON2                                                     BESJ402
      RTARY=RTZ*FN13                                                    BESJ403
      ARY=RTARY*RTARY                                                   BESJ404
```

Figure A3. (Sheet 10 of 25)

```
  150 CONTINUE                                                          BESJ405
      PHI=SQRT((RTZ+RTZ)*TAU)                                           BESJ406
      TB=1.                                                             BESJ407
      ASUM=1.                                                           BESJ408
      TFN=TAU/FN                                                        BESJ409
      UPOL(2)=(C(1,1)*T2+C(2,1))*TFN                                    BESJ410
      RCZ=CON1/CZ                                                       BESJ411
      CRZ32=CON548*RCZ                                                  BESJ412
      BSUM=UPOL(2)+CRZ32                                                BESJ413
      RELB=TOL*ABS(BSUM)                                                BESJ414
      AP=TFN                                                            BESJ415
      KS=0                                                              BESJ416
      KP1=2                                                             BESJ417
      RZDEN=RCZ                                                         BESJ418
      DO 155 LR=2,8,2                                                   BESJ419
C                                                                       BESJ420
C     COMPUTE TWO U POLYNOMIALS FOR NEXT A(ZETA) AND B(ZETA)            BESJ421
C                                                                       BESJ422
      LRP1=LR+1                                                         BESJ423
      DO 101 K=LR,LRP1                                                  BESJ424
      KS=KS+1                                                           BESJ425
      KP1=KP1+1                                                         BESJ426
      S1=C(1,K)                                                         BESJ427
      DO 102 J=2,KP1                                                    BESJ428
      S1=S1*T2+C(J,K)                                                   BESJ429
  102 CONTINUE                                                          BESJ430
      AP=AP*TFN                                                         BESJ431
      UPOL(KP1)=AP*S1                                                   BESJ432
      CR(KS)=BR(KS)*RZDEN                                               BESJ433
      RZDEN=RZDEN*RCZ                                                   BESJ434
      DR(KS)=AR(KS)*RZDEN                                               BESJ435
  101 CONTINUE                                                          BESJ436
      SUMA=UPOL(LRP1)                                                   BESJ437
      SUMB=UPOL(LR+2)+UPOL(LRP1)*CRZ32                                  BESJ438
      JU=LRP1                                                           BESJ439
      DO 151 JR=1,LR                                                    BESJ440
      JU=JU-1                                                           BESJ441
      SUMA=SUMA+CR(JR)*UPOL(JU)                                         BESJ442
      SUMB=SUMB+DR(JR)*UPOL(JU)                                         BESJ443
  151 CONTINUE                                                          BESJ444
      TB=-TB                                                            BESJ445
      IF(W2.GT.0.) TB=ABS(TB)                                           BESJ446
      ASUM=ASUM+SUMA*TB                                                 BESJ447
      BSUM=BSUM+SUMB*TB                                                 BESJ448
      IF(ABS(SUMA).LE.TOL.AND.ABS(SUMB).LE.RELB) GO TO 165              BESJ449
  155 CONTINUE                                                          BESJ450
  165 TB=RTARY                                                          BESJ451
      IF(W2.GT.0.) TB=-TB                                               BESJ452
      BSUM=BSUM/TB                                                      BESJ453
C                                                                       BESJ454
  400 CONTINUE                                                          BESJ455
      CALL JAIRY(ARY,RTARY,ACZ,AI,DAI)                                  BESJ456
      TEMP(IS)=PHI*(AI*ASUM+DAI*BSUM)/FN13                              BESJ457
      GO TO (401,202,650), IS                                           BESJ458
  402 TEMP(1)=TEMP(3)                                                   BESJ459
```

Figure A3. (Sheet 11 of 25)

```
      KT=1                                                          BESJ460
  401 IS=2                                                          BESJ461
      DFN=DFN-1.D+0                                                 BESJ462
      FN=DFN                                                        BESJ463
      GO TO 130                                                     BESJ464
C                                                                   BESJ465
C     SERIES FOR (X/2)**2.LE.NU+1                                   BESJ466
C                                                                   BESJ467
  134 CONTINUE                                                      BESJ468
      GLN=GAMLN(FNP1)                                               BESJ469
      ARG=FN*XO2L-GLN                                               BESJ470
      IF(ARG.LT.-ELIM1) GO TO 123                                   BESJ471
      EARG=EXP(ARG)                                                 BESJ472
  300 CONTINUE                                                      BESJ473
      S=1.                                                          BESJ474
      AK=3.                                                         BESJ475
      T2=1.                                                         BESJ476
      T=1.                                                          BESJ477
      S1=FN                                                         BESJ478
      DO 125 K=1,17                                                 BESJ479
      S2=T2+S1                                                      BESJ480
      T=-T*SXO2/S2                                                  BESJ481
      S=S+T                                                         BESJ482
      IF(ABS(T).LT.TOL) GO TO 127                                   BESJ483
      T2=T2+AK                                                      BESJ484
      AK=AK+2.                                                      BESJ485
      S1=S1+FN                                                      BESJ486
  125 CONTINUE                                                      BESJ487
  127 CONTINUE                                                      BESJ488
      TEMP(IS)=S*EARG                                               BESJ489
      GO TO (301,202,600), IS                                       BESJ490
  301 EARG=EARG*FN/XO2                                              BESJ491
      DFN=DFN-1.D+0                                                 BESJ492
      FN=DFN                                                        BESJ493
      IS=2                                                          BESJ494
      GO TO 300                                                     BESJ495
C                                                                   BESJ496
C     SET UNDERFLOW VALUE AND UPDATE PARAMETERS                     BESJ497
C                                                                   BESJ498
  180 Y(NN)=0.                                                      BESJ499
      NN=NN-1                                                       BESJ500
      DFN=DFN-1.D+0                                                 BESJ501
      FN=DFN                                                        BESJ502
      IF (NN-1) 170,171,130                                         BESJ 503
  171 KT=2                                                          BESJ504
      IS=2                                                          BESJ505
      GO TO 130                                                     BESJ506
  123 Y(NN)=0.                                                      BESJ507
      NN=NN-1                                                       BESJ508
      FNP1=FN                                                       BESJ509
      DFN=DFN-1.D+0                                                 BESJ510
      FN=DFN                                                        BESJ511
      IF(NN-1) 170,172,173                                          BESJ512
  172 KT=2                                                          BESJ513
      IS=2                                                          BESJ514
```

Figure A3. (Sheet 12 of 25)

```
  173 IF(SXO2.LE.FNP1) GO TO 133                                        BESJ515
      GO TO 130                                                         BESJ516
  133 ARG=ARG-XO2L+ALOG(FNP1)                                           BESJ517
      IF(ARG.LT.-ELIM1) GO TO 123                                       BESJ518
      GO TO 134                                                         BESJ519
  170 NZ=N-NN                                                           BESJ520
      RETURN                                                            BESJ521
C                                                                       BESJ522
C     BACKWARD RECURSION SECTION                                        BESJ523
C                                                                       BESJ524
  202 CONTINUE                                                          BESJ525
      NZ=N-NN                                                           BESJ526
      IF(KT.EQ.2) GO TO 250                                             BESJ527
  203 CONTINUE                                                          BESJ528
C     BACKWARD RECUR FROM INDEX ALPHA+NN-1 TO ALPHA                     BESJ529
      Y(NN)=TEMP(1)                                                     BESJ530
      Y(NN-1)=TEMP(2)                                                   BESJ531
      IF(NN.EQ.2) RETURN                                                BESJ532
      DX=X                                                              BESJ533
      TRX=2.D+0/DX                                                      BESJ534
      DTM=DFN*TRX                                                       BESJ535
      TM=DTM                                                            BESJ536
      K=NN+1                                                            BESJ537
      DO 230 I=3,NN                                                     BESJ538
      K=K-1                                                             BESJ539
      Y(K-2)=TM*Y(K-1)-Y(K)                                             BESJ540
      DTM=DTM-TRX                                                       BESJ541
      TM=DTM                                                            BESJ542
  230 CONTINUE                                                          BESJ543
      RETURN                                                            BESJ544
  250 Y(1)=TEMP(2)                                                      BESJ545
      RETURN                                                            BESJ546
C                                                                       BESJ547
C     ASYMPTOTIC EXPANSION FOR X TO INFINITY WITH FORWARD RECURSION IN  BESJ548
C     OSCILLATORY REGION X.GT.MAX(20, NU), PROVIDED THE LAST MEMBER     BESJ549
C     OF THE SEQUENCE IS ALSO IN THE REGION.                            BESJ550
C                                                                       BESJ551
  500 CONTINUE                                                          BESJ552
      IN=ALPHA-TAU+2.                                                   BESJ553
      IF(IN.LE.0) GO TO 502                                             BESJ554
      INP1=IN+1                                                         BESJ555
      DALPHA=ALPHA-FLOAT(INP1)                                          BESJ556
      KT=1                                                              BESJ557
      GO TO 511                                                         BESJ558
  502 DALPHA=ALPHA                                                      BESJ559
      IN=0                                                              BESJ560
  511 IS=KT                                                             BESJ561
  512 ARG=X-PIDT*DALPHA-PDF                                             BESJ562
      SA=SIN(ARG)                                                       BESJ563
      SB=COS(ARG)                                                       BESJ564
      RA=RTTP/RTX                                                       BESJ565
      ETX=8.*X                                                          BESJ566
  503 DX=DALPHA                                                         BESJ567
      DX=DX+DX                                                          BESJ568
      DTM=DX*DX                                                         BESJ569
```

Figure A3. (Sheet 13 of 25)

```
      T2=DTM-1.D+0                                                      BESJ570
      T2=T2/ETX                                                         BESJ571
      S2=T2                                                             BESJ572
      RELB=TOL*ABS(T2)                                                  BESJ573
      T1=ETX                                                            BESJ574
      S1=1.                                                             BESJ575
      FN=1.                                                             BESJ576
      AK=8.                                                             BESJ577
      DO 504 K=1,13                                                     BESJ578
      T1=T1+ETX                                                         BESJ579
      FN=FN+AK                                                          BESJ580
      DX=FN                                                             BESJ581
      TRX=DTM-DX                                                        BESJ582
      AP=TRX                                                            BESJ583
      T2=-T2*AP/T1                                                      BESJ584
      S1=S1+T2                                                          BESJ585
      T1=T1+ETX                                                         BESJ586
      AK=AK+8.                                                          BESJ587
      FN=FN+AK                                                          BESJ588
      DX=FN                                                             BESJ589
      TRX=DTM-DX                                                        BESJ590
      AP=TRX                                                            BESJ591
      T2= T2*AP/T1                                                      BESJ592
      S2=S2+T2                                                          BESJ593
      IF(ABS(T2).LE.RELB) GO TO 505                                     BESJ594
      AK=AK+8.                                                          BESJ595
  504 CONTINUE                                                          BESJ596
  505 TEMP(IS)=RA*(S1*SB-S2*SA)                                         BESJ597
      GO TO (506,507),IS                                                BESJ598
  506 DALPHA=DALPHA+1.                                                  BESJ599
      IS=2                                                              BESJ600
      TB=SA                                                             BESJ601
      SA=-SB                                                            BESJ602
      SB=TB                                                             BESJ603
      GO TO 503                                                         BESJ604
C                                                                       BESJ605
C     FORWARD RECURSION SECTION                                         BESJ606
C                                                                       BESJ607
  507 IF(KT.EQ.2) GO TO 250                                             BESJ608
      S1=TEMP(1)                                                        BESJ609
      S2=TEMP(2)                                                        BESJ610
      TX=2./X                                                           BESJ611
      TM=DALPHA*TX                                                      BESJ612
      IF(IN.EQ.0) GO TO 520                                             BESJ613
C                                                                       BESJ614
C     FORWARD RECUR TO INDEX ALPHA                                      BESJ615
C                                                                       BESJ616
      DO 510 I=1,IN                                                     BESJ617
      S=S2                                                              BESJ618
      S2=TM*S2-S1                                                       BESJ619
      TM=TM+TX                                                          BESJ620
      S1=S                                                              BESJ621
  510 CONTINUE                                                          BESJ622
      IF(NN.EQ.1) GO TO 535                                             BESJ623
      S=S2                                                              BESJ624
```

Figure A3. (Sheet 14 of 25)

```
      S2=TM*S2-S1                                                       BESJ625
      TM=TM+TX                                                          BESJ626
      S1=S                                                              BESJ627
  520 CONTINUE                                                          BESJ628
C                                                                       BESJ629
C     FORWARD RECUR FROM INDEX ALPHA TO ALPHA+N-1                       BESJ630
C                                                                       BESJ631
      Y(1)=S1                                                           BESJ632
      Y(2)=S2                                                           BESJ633
      IF(NN.EQ.2) RETURN                                                BESJ634
      DO 530 I=3,NN                                                     BESJ635
      Y(I)=TM*Y(I-1)-Y(I-2)                                             BESJ636
      TM=TM+TX                                                          BESJ637
  530 CONTINUE                                                          BESJ638
      RETURN                                                            BESJ639
  535 Y(1)=S2                                                           BESJ640
      RETURN                                                            BESJ641
C                                                                       BESJ642
C     BACKWARD RECURSION WITH NORMALIZATION BY                          BESJ643
C     ASYMPTOTIC EXPANSION FOR NU TO INFINITY OR POWER SERIES.          BESJ644
C                                                                       BESJ645
  600 CONTINUE                                                          BESJ646
C     COMPUTATION OF LAST ORDER FOR SERIES NORMALIZATION                BESJ647
      KM=AMAX1(3.-FN,0.)                                                BESJ648
      TFN=FN+FLOAT(KM)                                                  BESJ649
      TA=(GLN+TFN-0.9189385332-0.0833333333/TFN)/(TFN+0.5)              BESJ650
      TA=XO2L-TA                                                        BESJ651
      TB=-(1.-1.5/TFN)/TFN                                              BESJ652
      IN=CE/(-TA+SQRT(TA*TA-CE*TB))+1.5                                 BESJ653
      IN=IN+KM                                                          BESJ654
      GO TO 603                                                         BESJ655
  650 CONTINUE                                                          BESJ656
C     COMPUTATION OF LAST ORDER FOR ASYMPTOTIC EXPANSION NORMALIZATION  BESJ657
      GLN=AZ32+RA                                                       BESJ658
      IF(ARY.GT.30.) GO TO 675                                          BESJ659
      RDEN=(PP(4)*ARY+PP(3))*ARY+1.                                     BESJ660
      RZDEN=PP(1)+PP(2)*ARY                                             BESJ661
      TA=RZDEN/RDEN                                                     BESJ662
      IF(W2.LT.0.10) GO TO 651                                          BESJ663
      TB=GLN/RTARY                                                      BESJ664
      GO TO 677                                                         BESJ665
  651 TB=(1.259921049+0.1679894730*W2)/FN13                             BESJ666
      GO TO 677                                                         BESJ667
  675 CONTINUE                                                          BESJ668
      TA=CON1*TCE/ACZ                                                   BESJ669
      TA=((0.0493827160*TA-0.1111111111)*TA+0.6666666667)*TA*ARY        BESJ670
      IF(W2.LT.0.10) GO TO 651                                          BESJ671
      TB=GLN/RTARY                                                      BESJ672
  677 IN=TA/TB+1.5                                                      BESJ673
      IF(IN.GT.INLIM) GO TO 402                                         BESJ674
  603 DX=FLOAT(IN)                                                      BESJ675
      DTM=DFN+DX                                                        BESJ676
      DX=X                                                              BESJ677
      TRX=2.D+0/DX                                                      BESJ678
      DTM=DTM*TRX                                                       BESJ679
```

Figure A3. (Sheet 15 of 25)

```
      TM=DTM                                                        BESJ680
      TA=0.                                                         BESJ681
      TB=TOL                                                        BESJ682
      KK=1                                                          BESJ683
  605 CONTINUE                                                      BESJ684
C                                                                   BESJ685
C     BACKWARD RECUR UNINDEXED                                      BESJ686
C                                                                   BESJ687
      DO 601 I=1,IN                                                 BESJ688
      S=TB                                                          BESJ689
      TB=TM*TB-TA                                                   BESJ690
      TA=S                                                          BESJ691
      DTM=DTM-TRX                                                   BESJ692
      TM=DTM                                                        BESJ693
  601 CONTINUE                                                      BESJ694
C     NORMALIZATION                                                 BESJ695
      IF(KK.NE.1) GO TO 604                                         BESJ696
      TA=(TA/TB)*TEMP(3)                                            BESJ697
      TB=TEMP(3)                                                    BESJ698
      KK=2                                                          BESJ699
      IN=NS                                                         BESJ700
      IF(NS.NE.0) GO TO 605                                         BESJ701
  604 Y(NN)=TB                                                      BESJ702
  615 NZ=N-NN                                                       BESJ703
      IF(NN.EQ.1) RETURN                                            BESJ704
      S=TB                                                          BESJ705
      TB=TM*TB-TA                                                   BESJ706
      TA=S                                                          BESJ707
      DTM=DTM-TRX                                                   BESJ708
      TM=DTM                                                        BESJ709
      K=NN-1                                                        BESJ710
      Y(K)=TB                                                       BESJ711
      IF(NN.EQ.2) RETURN                                            BESJ712
      KM=K-1                                                        BESJ713
C                                                                   BESJ714
C     BACKWARD RECUR INDEXED                                        BESJ715
C                                                                   BESJ716
      DO 602 I=1,KM                                                 BESJ717
      Y(K-1)=TM*Y(K)-Y(K+1)                                         BESJ718
      DTM=DTM-TRX                                                   BESJ719
      TM=DTM                                                        BESJ720
      K=K-1                                                         BESJ721
  602 CONTINUE                                                      BESJ722
      RETURN                                                        BESJ723
C                                                                   BESJ724
C                                                                   BESJ725
C                                                                   BESJ726
   91 CONTINUE                                                      BESJ727
      NZ=-2                                                         BESJ728
      RETURN                                                        BESJ729
   92 CONTINUE                                                      BESJ730
      NZ=-3                                                         BESJ731
      RETURN                                                        BESJ732
   93 CONTINUE                                                      BESJ733
      NZ=-1                                                         BESJ734
```

Figure A3. (Sheet 16 of 25)

```
      RETURN                                                            BESJ735
      END                                                               BESJ736
      SUBROUTINE JAIRY(X,RX,C,AI,DAI)                                   AIRY1
C  - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - -AIRY2
C     CDC 6600 ROUTINE   1-2-74                                         AIRY3
C                  JAIRY COMPUTES THE AIRY FUNCTION AI(X)               AIRY4
C                   AND ITS DERIVATIVE DAI(X) FOR JBESS                 AIRY5
C                                                                       AIRY6
C     INPUT:  X - ARGUMENT, COMPUTED BY JBESS, X UNRESTRICTED           AIRY7
C             RX - RX=SQRT(ABS(X)), COMPUTED BY JBESS                   AIRY8
C             C - C=2.*(ABS(X)**1.5)/3., COMPUTED BY JBESS              AIRY9
C     OUTPUT: AI - VALUE OF FUNCTION AI(X)                              AIRY10
C             DAI - VALUE OF THE DERIVATIVE DAI(X)                      AIRY11
C                                                                       AIRY12
C      WRITTEN BY D.E. AMOS, S.L. DANIEL & M.K WESTON                   AIRY13
C  - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - -AIRY14
      DIMENSION AK1(14),AK2(23),AK3(14)                                 AIRY15
      DIMENSION AJP(19),AJN(19),A(15),B(15)                             AIRY16
      DIMENSION DAK1(14),DAK2(24),DAK3(14)                              AIRY17
      DIMENSION DAJP(19),DAJN(19),DA(15),DB(15)                         AIRY18
      DATA N1,N2,N3,N4/14,23,19,15/                                     AIRY19
      DATA M1,M2,M3,M4/12,21,17,13/                                     AIRY20
      DATA FPI12,CON1,CON2,CON3,CON4,CON5/                              AIRY21
     1 1.3089969389958E+00, 6.6666666666667E-01, 5.0315471619678E+00,   AIRY22
     2 3.8000458986729E-01, 8.3333333333333E-01, 8.6602540378444E-01/   AIRY23
      DATA AK1(1) / 2.2042309098779E-01/,                               AIRY24
     1 AK1(2) /-1.2529024278770E-01/, AK1(3) / 1.0388116335919E-02/,    AIRY25
     2 AK1(4) / 8.2284415200634E-04/, AK1(5) /-2.3461434589123E-04/,    AIRY26
     3 AK1(6) / 1.6382428017212E-05/, AK1(7) / 3.0690258957319E-07/,    AIRY27
     4 AK1(8) /-1.2962199935933E-07/, AK1(9) / 8.2290815882367E-09/,    AIRY28
     5 AK1(10)/ 1.5396396862330E-11/, AK1(11)/-3.3916546561568E-11/,    AIRY29
     6 AK1(12)/ 2.0325325742363E-12/, AK1(13)/-1.1067954609788E-14/,    AIRY30
     7 AK1(14)/-5.1616949778508E-15/                                    AIRY31
      DATA AK2(1) / 2.7436615086960E-01/,                               AIRY32
     1 AK2(2) / 5.3979096973690E-03/, AK2(3) /-1.5733922062119E-03/,    AIRY33
     2 AK2(4) / 4.2742752824875E-04/, AK2(5) /-1.1212491739992E-04/,    AIRY34
     3 AK2(6) / 2.8876317131890E-05/, AK2(7) /-7.3680422537055E-06/,    AIRY35
     4 AK2(8) / 1.8729020974102E-06/, AK2(9) /-4.7589279396229E-07/,    AIRY36
     5 AK2(10)/ 1.2113041695591E-07/, AK2(11)/-3.0924537427061E-08/,    AIRY37
     6 AK2(12)/ 7.9245470528265E-09/, AK2(13)/-2.0390244716791E-09/,    AIRY38
     7 AK2(14)/ 5.2686305659574E-10/, AK2(15)/-1.3670476763957E-10/,    AIRY39
     8 AK2(16)/ 3.5614103901371E-11/, AK2(17)/-9.3138829654843E-12/,    AIRY40
     9 AK2(18)/ 2.4446445047364E-12/, AK2(19)/-6.4384026199096E-13/,    AIRY41
     1 AK2(20)/ 1.7010603055935E-13/, AK2(21)/-4.5076010450328E-14/,    AIRY42
     2 AK2(22)/ 1.1977479916481E-14/, AK2(23)/-3.1907704086507E-15/     AIRY43
      DATA AK3(1) / 2.8027144734079E-01/,                               AIRY44
     1 AK3(2) /-1.7812704284438E-03/, AK3(3) / 4.0342257962900E-05/,    AIRY45
     2 AK3(4) /-1.6324996526900E-06/, AK3(5) / 9.2118148247677E-08/,    AIRY46
     3 AK3(6) /-6.5229433022916E-09/, AK3(7) / 5.4713840457655E-10/,    AIRY47
     4 AK3(8) /-5.2440825180026E-11/, AK3(9) / 5.6047790411721E-12/,    AIRY48
     5 AK3(10)/-6.5637524463931E-13/, AK3(11)/ 8.3128576196625E-14/,    AIRY49
     6 AK3(12)/-1.1270513469106E-14/, AK3(13)/ 1.6226797659813E-15/,    AIRY50
     7 AK3(14)/-2.4648032431243E-16/                                    AIRY51
      DATA AJP(1) / 7.7895296643758E-02/,                               AIRY52
     1 AJP(2) /-1.8435636345680E-01/, AJP(3) / 3.0141260521617E-02/,    AIRY53
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2 AJP(4) / 3.0534272427761E-02/, AJP(5) /-4.9542470251308E-03/,    AIRY54
3 AJP(6) /-1.7274955256395E-03/, AJP(7) / 2.4313763783919E-04/,    AIRY55
4 AJP(8) / 5.0456477751708E-05/, AJP(9) /-6.1631658269521E-06/,    AIRY56
5 AJP(10)/-9.0398674551077E-07/, AJP(11)/ 9.7024377835588E-08/,    AIRY57
6 AJP(12)/ 1.0963945330520E-08/, AJP(13)/-1.0471633058877E-09/,    AIRY58
7 AJP(14)/-9.6035944134465E-11/, AJP(15)/ 8.2535878945413E-12/,    AIRY59
8 AJP(16)/ 6.3612343901877E-13/, AJP(17)/-4.9662961411602E-14/,    AIRY60
9 AJP(18)/-3.2981028892962E-15/, AJP(19)/ 2.3579825203110E-16/     AIRY61
 DATA AJN(1) / 3.8049788761724E-02/,                               AIRY62
1 AJN(2) /-2.4531954184555E-01/, AJN(3) / 1.6582062370270E-01/,    AIRY63
2 AJN(4) / 7.4933004581879E-02/, AJN(5) /-2.6347628810664E-02/,    AIRY64
3 AJN(6) /-5.9253559730498E-03/, AJN(7) / 1.4474440958980E-03/,    AIRY65
4 AJN(8) / 2.1831183132222E-04/, AJN(9) /-4.1066207768030E-05/,    AIRY66
5 AJN(10)/-4.6687499417177E-06/, AJN(11)/ 7.1521880727716E-07/,    AIRY67
6 AJN(12)/ 6.5296477085463E-08/, AJN(13)/-8.4428402756595E-09/,    AIRY68
7 AJN(14)/-6.4418615897698E-10/, AJN(15)/ 7.2080228650528E-11/,    AIRY69
8 AJN(16)/ 4.7246543171785E-12/, AJN(17)/-4.6602283254704E-13/,    AIRY70
9 AJN(18)/-2.6776271038919E-14/, AJN(19)/ 2.3616131657002E-15/     AIRY71
 DATA A(1) / 4.9027542474279E-01/, A(2) / 1.5764727794620E-03/,    AIRY72
1     A(3) /-9.6619596314031E-05/, A(4) / 1.3591608026882E-07/,    AIRY73
2     A(5) / 2.9815734265486E-07/, A(6) /-1.8682476755998E-08/,    AIRY74
3     A(7) /-1.0368573766714E-09/, A(8) / 3.2866081843433E-10/,    AIRY75
4     A(9) /-2.5709141063278E-11/, A(10)/-2.3235765530068E-12/,    AIRY76
5     A(11)/ 9.5752327904826E-13/, A(12)/-1.2034082804972E-13/,    AIRY77
6     A(13)/-2.9090771677072E-15/, A(14)/ 4.5565645458015E-15/,    AIRY78
7     A(15)/-9.9900387481026E-16/                                  AIRY79
 DATA B(1) / 2.785935528030BE-01/, B(2) /-3.5291569188258E-03/,    AIRY80
1     B(3) /-2.3114967738499E-05/, B(4) / 4.7131784226356E-06/,    AIRY81
2     B(5) /-1.1241590793133E-07/, B(6) /-2.0010030118434E-08/,    AIRY82
3     B(7) / 2.6094807530219E-09/, B(8) /-3.5509813610122E-11/,    AIRY83
4     B(9) /-3.5084997842388E-11/, B(10)/ 5.8300718795420E-12/,    AIRY84
5     B(11)/-2.0464482875333E-13/, B(12)/-1.1052917947674E-13/,    AIRY85
6     B(13)/ 2.8772477803878E-14/, B(14)/-2.8820511100994E-15/,    AIRY86
7     B(15)/-3.3265631169617E-16/                                  AIRY87
 DATA N1D,N2D,N3D,N4D/14,24,19,15/                                 AIRY88
 DATA M1D,M2D,M3D,M4D/12,22,17,13/                                 AIRY89
 DATA DAK1(1) / 2.0456784230789E-01/,                              AIRY90
1 DAK1(2) /-6.6132273990566E-02/, DAK1(3) /-8.4984580098929E-03/,  AIRY91
2 DAK1(4) / 3.1218349155629E-03/, DAK1(5) /-2.7001648982943E-04/,  AIRY92
3 DAK1(6) /-6.3563629867939E-06/, DAK1(7) / 3.0239771240951E-06/,  AIRY93
4 DAK1(8) /-2.1831119533009E-07/, DAK1(9) /-5.3619428933283E-10/,  AIRY94
5 DAK1(10)/ 1.1309803562231E-09/, DAK1(11)/-7.4302383462907E-11/,  AIRY95
6 DAK1(12)/ 4.2880417082689E-13/, DAK1(13)/ 2.2381092575454E-13/,  AIRY96
7 DAK1(14)/-1.3914013564118E-14/                                   AIRY97
 DATA DAK2(1) / 2.9333234388323E-01/,                              AIRY98
1 DAK2(2) /-8.0619678474311E-03/, DAK2(3) / 2.4254017233314E-03/,  AIRY99
2 DAK2(4) /-6.8229754885024E-04/, DAK2(5) / 1.8578642775118E-04/,  AIRY100
3 DAK2(6) /-4.9745744768406E-05/, DAK2(7) / 1.3209068123950E-05/,  AIRY101
4 DAK2(8) /-3.4952824044494E-06/, DAK2(9) / 9.2436245107884E-07/,  AIRY102
5 DAK2(10)/-2.4473267152187E-07/, DAK2(11)/ 6.4930783764891E-08/,  AIRY103
6 DAK2(12)/-1.7271762150154E-08/, DAK2(13)/ 4.6072576360466E-09/,  AIRY104
7 DAK2(14)/-1.2324905529155E-09/, DAK2(15)/ 3.3062040948810E-10/,  AIRY105
8 DAK2(16)/-8.8925209977240E-11/, DAK2(17)/ 2.3977331987830E-11/,  AIRY106
9 DAK2(18)/-6.4801392115345E-12/, DAK2(19)/ 1.7551013202373E-12/,  AIRY107
1 DAK2(20)/-4.7630382983364E-13/, DAK2(21)/ 1.2949824110081E-13/,  AIRY108
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     2 DAK2(22)/-3.5267962221043E-14/, DAK2(23)/ 9.6200515158592E-15/,  AIRY109
     3 DAK2(24)/-2.6278691434229E-15/                                   AIRY110
      DATA DAK3(1) / 2.8467582881135E-01/,                              AIRY111
     1 DAK3(2) / 2.5307307261908E-03/, DAK3(3) /-4.8348113033798E-05/,  AIRY112
     2 DAK3(4) / 1.8490728394634E-06/, DAK3(5) /-1.0141849117858E-07/,  AIRY113
     3 DAK3(6) / 7.0592563445715E-09/, DAK3(7) /-5.8532529140038E-10/,  AIRY114
     4 DAK3(8) / 5.5635768883134E-11/, DAK3(9) /-5.9088909477950E-12/,  AIRY115
     5 DAK3(10)/ 6.8857435378444E-13/, DAK3(11)/-8.6858825645219E-14/,  AIRY116
     6 DAK3(12)/ 1.1737476261721E-14/, DAK3(13)/-1.6852314651092E-15/,  AIRY117
     7 DAK3(14)/ 2.5537477309706E-16/                                   AIRY118
      DATA DAJP(1) / 6.5321913131146E-02/,                              AIRY119
     1 DAJP(2) /-1.2026293368882E-01/, DAJP(3) / 9.7801023626382E-03/,  AIRY120
     2 DAJP(4) / 1.6794842923050E-02/, DAJP(5) /-1.9714614018213E-03/,  AIRY121
     3 DAJP(6) /-8.4556029509887E-04/, DAJP(7) / 9.4288962070198E-05/,  AIRY122
     4 DAJP(8) / 2.2582786094548E-05/, DAJP(9) /-2.2906787091599E-06/,  AIRY123
     5 DAJP(10)/-3.7834399113692E-07/, DAJP(11)/ 3.4566393355956E-08/,  AIRY124
     6 DAJP(12)/ 4.2961133200301E-09/, DAJP(13)/-3.5867369121499E-10/,  AIRY125
     7 DAJP(14)/-3.5724588136190E-11/, DAJP(15)/ 2.7269609106634E-12/,  AIRY126
     8 DAJP(16)/ 2.2612065309577E-13/, DAJP(17)/-1.5876320523830E-14/,  AIRY127
     9 DAJP(18)/-1.1260437448512E-15/, DAJP(19)/ 7.3132752951537E-17/   AIRY128
      DATA DAJN(1) / 1.0859453963297E-02/,                              AIRY129
     1 DAJN(2) / 8.5331319485709E-02/, DAJN(3) /-3.1527706811306E-01/,  AIRY130
     2 DAJN(4) /-8.7842072529426E-02/, DAJN(5) / 5.5325190697605E-02/,  AIRY131
     3 DAJN(6) / 9.4167406050324E-03/, DAJN(7) /-3.3218702601900E-03/,  BEJS132
     4 DAJN(8) /-4.1115734315683E-04/, DAJN(9) / 1.0129732689135E-04/,  AIRY133
     5 DAJN(10)/ 9.8763368220840E-06/, DAJN(11)/-1.8731296981239E-06/,  AIRY134
     6 DAJN(12)/-1.5079850013147E-07/, DAJN(13)/ 2.3268766952539E-08/,  AIRY135
     7 DAJN(14)/ 1.5959991741922E-09/, DAJN(15)/-2.0766592266838E-10/,  AIRY136
     8 DAJN(16)/-1.2410335050030E-11/, DAJN(17)/ 1.3963176533104E-12/,  AIRY137
     9 DAJN(18)/ 7.3940097115574E-14/, DAJN(19)/-7.3288747562750E-15/   AIRY137
      DATA DA(1) / 4.9162732110460E-01/, DA(2) / 3.1116493042749E-03/,  AIRY139
     1     DA(3) / 8.2314076285408E-05/, DA(4) /-4.6176977617214E-06/,  AIRY140
     2     DA(5) /-6.1315888053463E-08/, DA(6) / 2.8729580465652E-08/,  AIRY141
     3     DA(7) /-1.8195971537212E-09/, DA(8) /-1.4475282664204E-10/,  AIRY142
     4     DA(9) / 4.5372404342042E-11/, DA(10)/-3.9965506584722E-12/,  AIRY143
     5     DA(11)/-3.2408911983032E-13/, DA(12)/ 1.6209895256874E-13/,  AIRY144
     6     DA(13)/-2.4076524797406E-14/, DA(14)/ 1.6938481128449E-16/,  AIRY145
     7     DA(15)/ 8.1790078647740E-16/                                 AIRY146
      DATA DB(1) /-2.7757135694423E-01/, DB(2) / 4.4421283341992E-03/,  AIRY147
     1     DB(3) /-8.4232852219009E-05/, DB(4) /-2.5804031841871E-06/,  AIRY148
     2     DB(5) / 3.4238972021762E-07/, DB(6) /-6.2428689470978E-09/,  AIRY149
     3     DB(7) /-2.3637783684458E-09/, DB(8) / 3.1699104265667E-10/,  AIRY150
     4     DB(9) /-4.4099569165819E-12/, DB(10)/-5.1867422109358E-12/,  AIRY151
     5     DB(11)/ 9.6487401513702E-13/, DB(12)/-4.9019057660871E-14/,  AIRY152
     6     DB(13)/-1.7725343067811E-14/, DB(14)/ 5.5595061044266E-15/,  AIRY153
     7     DB(15)/-7.1179333757953E-16/                                 AIRY154
C     -------------------                                               AIRY155
      IF(X.LT.0.) GO TO 300                                             AIRY156
      IF(C.GT.5.) GO TO 200                                             AIRY157
      IF(X.GT.1.2) GO TO 150                                            AIRY158
      T=(X+X-1.2)*CON4                                                  AIRY159
      TT = T + T                                                        AIRY160
      J=N1                                                              AIRY161
      F1=AK1(J)                                                         AIRY162
      F2=0.                                                             AIRY163
```

Figure A3. (Sheet 19 of 25)

```
      DO 105 I=1,M1                                                     AIRY164
      J=J-1                                                             AIRY165
      TEMP1=F1                                                          AIRY166
      F1=TT*F1-F2+AK1(J)                                                AIRY167
      F2=TEMP1                                                          AIRY168
  105 CONTINUE                                                          AIRY169
      AI=T*F1-F2+AK1(1)                                                 AIRY170
C                                                                       AIRY171
      J=N1D                                                             AIRY172
      F1=DAK1(J)                                                        AIRY173
      F2=0.                                                             AIRY174
      DO 106 I=1,M1D                                                    AIRY175
      J=J-1                                                             AIRY176
      TEMP1=F1                                                          AIRY177
      F1=TT*F1-F2+DAK1(J)                                               AIRY178
      F2=TEMP1                                                          AIRY179
106   CONTINUE                                                          AIRY180
      DAI=-(T*F1-F2+DAK1(1))                                            AIRY181
      RETURN                                                            AIRY182
C                                                                       AIRY183
  150 CONTINUE                                                          AIRY184
      T=(X+X-CON2)*CON3                                                 AIRY185
      TT = T + T                                                        AIRY186
      J=N2                                                              AIRY187
      F1=AK2(J)                                                         AIRY188
      F2=0.                                                             AIRY189
      DO 155 I=1,M2                                                     AIRY190
      J=J-1                                                             AIRY191
      TEMP1=F1                                                          AIRY192
      F1=TT*F1-F2+AK2(J)                                                AIRY193
      F2=TEMP1                                                          AIRY194
  155 CONTINUE                                                          AIRY195
      RTRX=SQRT(RX)                                                     AIRY196
      EC=EXP(-C)                                                        AIRY197
      AI=EC*(T*F1-F2+AK2(1))/RTRX                                       AIRY198
      J=N2D                                                             AIRY199
      F1=DAK2(J)                                                        AIRY200
      F2=0.                                                             AIRY201
      DO 156 I=1,M2D                                                    AIRY202
      J=J-1                                                             AIRY203
      TEMP1=F1                                                          AIRY204
      F1=TT*F1-F2+DAK2(J)                                               AIRY205
      F2=TEMP1                                                          AIRY206
 156  CONTINUE                                                          AIRY207
      DAI=-EC*(T*F1-F2+DAK2(1))*RTRX                                    AIRY208
      RETURN                                                            AIRY209
C                                                                       AIRY210
  200 CONTINUE                                                          AIRY211
      T=10./C-1.                                                        AIRY212
      TT=T+T                                                            AIRY213
      J=N1                                                              AIRY214
      F1=AK3(J)                                                         AIRY215
      F2=0.                                                             AIRY216
      DO 205 I=1,M1                                                     AIRY217
      J=J-1                                                             AIRY218
```

Figure A3. (Sheet 20 of 25)

```
      TEMP1=F1                                                          AIRY219
      F1=TT*F1-F2+AK3(J)                                                AIRY220
      F2=TEMP1                                                          AIRY221
  205 CONTINUE                                                          AIRY222
      RTRX=SQRT(RX)                                                     AIRY223
      EC=EXP(-C)                                                        AIRY224
      AI=EC*(T*F1-F2+AK3(1))/RTRX                                       AIRY225
      J=N1D                                                             AIRY226
      F1=DAK3(J)                                                        AIRY227
      F2=0.                                                             AIRY228
      DO 206 I=1,M1D                                                    AIRY229
      J=J-1                                                             AIRY230
      TEMP1=F1                                                          AIRY231
      F1=TT*F1-F2+DAK3(J)                                               AIRY232
      F2=TEMP1                                                          AIRY233
206   CONTINUE                                                          AIRY234
      DAI=-RTRX*EC*(T*F1-F2+DAK3(1))                                    AIRY235
      RETURN                                                            AIRY236
C                                                                       AIRY237
  300 CONTINUE                                                          AIRY238
      IF(C.GT.5.) GO TO 350                                             AIRY239
      T=.4*C-1.                                                         AIRY240
      TT=T+T                                                            AIRY241
      J=N3                                                              AIRY242
      F1=AJP(J)                                                         AIRY243
      E1=AJN(J)                                                         AIRY244
      F2=0.                                                             AIRY245
      E2=0.                                                             AIRY246
      DO 305 I=1,M3                                                     AIRY247
      J=J-1                                                             AIRY248
      TEMP1=F1                                                          AIRY249
      TEMP2=E1                                                          AIRY250
      F1=TT*F1-F2+AJP(J)                                                AIRY251
      E1=TT*E1-E2+AJN(J)                                                AIRY252
      F2=TEMP1                                                          AIRY253
      E2=TEMP2                                                          AIRY254
  305 CONTINUE                                                          AIRY255
      AI=(T*E1-E2+AJN(1))-X*(T*F1-F2+AJP(1))                            AIRY256
      J=N3D                                                             AIRY257
      F1=DAJP(J)                                                        AIRY258
      E1=DAJN(J)                                                        AIRY259
      F2=0.                                                             AIRY260
      E2=0.                                                             AIRY261
      DO 306 I=1,M3D                                                    AIRY262
      J=J-1                                                             AIRY263
      TEMP1=F1                                                          AIRY264
      TEMP2=E1                                                          AIRY265
      F1 = TT*F1-F2+DAJP(J)                                             AIRY266
      E1= TT*E1-E2+DAJN(J)                                              AIRY267
      F2=TEMP1                                                          AIRY268
      E2=TEMP2                                                          AIRY269
 306  CONTINUE                                                          AIRY270
      DAI=X*X*(T*F1-F2+DAJP(1))+(T*E1-E2+DAJN(1))                       AIRY271
      RETURN                                                            AIRY272
C                                                                       AIRY273
```

Figure A3. (Sheet 21 of 25)

```
  350 CONTINUE                                                          AIRY274
      T=10./C-1.                                                        AIRY275
      TT=T+T                                                            AIRY276
      J=N4                                                              AIRY277
      F1=A(J)                                                           AIRY278
      E1=B(J)                                                           AIRY279
      F2=0.                                                             AIRY280
      E2=0.                                                             AIRY281
      DO 310 I=1,M4                                                     AIRY282
      J=J-1                                                             AIRY283
      TEMP1=F1                                                          AIRY284
      TEMP2=E1                                                          AIRY285
      F1=TT*F1-F2+A(J)                                                  AIRY286
      E1=TT*E1-E2+B(J)                                                  AIRY287
      F2=TEMP1                                                          AIRY288
      E2=TEMP2                                                          AIRY289
  310 CONTINUE                                                          AIRY290
      TEMP1=T*F1-F2+A(1)                                                AIRY291
      TEMP2=T*E1-E2+B(1)                                                AIRY292
      RTRX=SQRT(RX)                                                     AIRY293
      CV=C-FPI12                                                        AIRY294
      CCV=COS(CV)                                                       AIRY295
      SCV=SIN(CV)                                                       AIRY296
      AI=(TEMP1*CCV-TEMP2*SCV)/RTRX                                     AIRY297
      J=N4D                                                             AIRY298
      F1=DA(J)                                                          AIRY299
      E1=DB(J)                                                          AIRY300
      F2=0.                                                             AIRY301
      E2=0.                                                             AIRY302
      DO 311 I=1,M4D                                                    AIRY303
      J=J-1                                                             AIRY304
      TEMP1=F1                                                          AIRY305
      TEMP2=E1                                                          AIRY306
      F1=TT*F1-F2+DA(J)                                                 AIRY307
      E1=TT*E1-E2+DB(J)                                                 AIRY308
      F2=TEMP1                                                          AIRY309
      E2=TEMP2                                                          AIRY310
  311 CONTINUE                                                          AIRY311
      TEMP1=T*F1-F2+DA(1)                                               AIRY312
      TEMP2=T*E1-E2+DB(1)                                               AIRY313
      E1=CCV*CON5+.5*SCV                                                AIRY314
      E2=SCV*CON5-.5*CCV                                                AIRY315
      DAI=(TEMP1*E1-TEMP2*E2)*RTRX                                      AIRY316
      RETURN                                                            AIRY317
      END                                                               AIRY318
      FUNCTION GAMLN(X)                                                 GLN1
C  - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - -GLN2
C     WRITTEN BY D. E. AMOS, SEPTEMBER, 1977.                          GLN3
C     REFERENCES                                                       GLN4
C      *  SAND-77-1518                                                 GLN5
C      *  COMPUTER APPROXIMATIONS BY J.F.HART, ET.AL., SIAM SERIES IN  GLN6
C         APPLIED MATHEMATICS, WILEY, 1968, P.135-136.                 GLN7
C      *  NBS HANDBOOK OF MATHEMATICAL FUNCTIONS, AMS 55, BY           GLN8
C         M. ABRAMOWITZ AND I.A. STEGUN, DECEMBER. 1955, P.257.        GLN9
C     ABSTRACT                                                         GLN10
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Figure A3. (Sheet 22 of 25)

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C         GAMLN COMPUTES THE NATURAL LOG OF THE GAMMA FUNCTION FOR      GLN11
C         X.GT.0. A RATIONAL CHEBYSHEV APPROXIMATION IS USED ON         GLN12
C         8.LT.X.LT.1000., THE ASYMPTOTIC EXPANSION FOR X.GE.1000. AND  GLN13
C         A RATIONAL CHEBYSHEV APPROXIMATION ON 2.LT.X.LT.3. FOR        GLN14
C         0.LT.X.LT.8. AND X NON-INTEGRAL, FORWARD OR BACKWARD          GLN15
C         RECURSION FILLS IN THE INTERVALS  0.LT.X.LT.2 AND             GLN16
C         3.LT.X.LT.8. FOR X=1.,2.,...,100., GAMLN IS SET TO            GLN17
C         NATURAL LOGS OF FACTORIALS.                                   GLN18
C                                                                       GLN19
C     DESCRIPTION OF ARGUMENTS                                          GLN20
C         INPUT                                                         GLN21
C           X       - X.GT.0                                            GLN22
C         OUTPUT                                                        GLN23
C           GAMLN  - NATURAL LOG OF THE GAMMA FUNCTION AT X             GLN24
C     ------------------                                                GLN25
      DIMENSION GLN(100),P(5),Q(2),PCOE(9),QCOE(4)                      GLN26
      DATA XLIM1,XLIM2,RTWPIL/    8.  ,  1000.   , 9.189385332047E-01/  GLN27
      DATA P(1)/7.663451880000E-04/,  P(2)/-5.940956105200E-04/,        GLN28
     1     P(3)/7.936431104845E-04/,  P(4)/-2.777777756577E-03/,        GLN29
     2     P(5)/8.333333333332E-02/                                     GLN30
      DATA Q(1)/-2.777777777778E-03/,  Q(2)/8.333333333333E-02/         GLN31
      DATA PCOE(1)/2.973786644810E-03/,PCOE(2)/9.238194559028E-03/,     GLN32
     1     PCOE(3)/1.093115956710E-01/,PCOE(4)/3.980671310204E-01/,     GLN33
     2     PCOE(5)/2.159943128461E+00/,PCOE(6)/6.338067999387E+00/,     GLN34
     3     PCOE(7)/2.078247253179E+01/,PCOE(8)/3.603677253002E+01/,     GLN35
     4     PCOE(9)/6.200383800713E+01/                                  GLN36
C                                                                       GLN37
      DATA QCOE(1)/1.000000000000E+00/,QCOE(2)/-8.906016659498E+00/,    GLN38
     1     QCOE(3)/9.822521104714E+00/,QCOE(4)/6.200383800713E+01/      GLN39
C                                                                       GLN40
      DATA GLN(1) /0.0/, GLN(2) /0.0/, GLN(3) /6.931471805599E-01/,     GLN41
     1     GLN(4) /1.791759469228E+00/, GLN(5) /3.178053830348E+00/,    GLN42
     2     GLN(6) /4.787491742782E+00/, GLN(7) /6.579251212010E+00/,    GLN43
     3     GLN(8) /8.525161361065E+00/, GLN(9) /1.060460290274E+01/,    GLN44
     4     GLN(10)/1.280182748008E+01/, GLN(11)/1.510441257308E+01/,    GLN45
     5     GLN(12)/1.750230784587E+01/, GLN(13)/1.998721449566E+01/,    GLN46
     6     GLN(14)/2.255216385312E+01/, GLN(15)/2.519122118274E+01/,    GLN47
     7     GLN(16)/2.789927138384E+01/, GLN(17)/3.067186010608E+01/,    GLN48
     8     GLN(18)/3.350507345014E+01/, GLN(19)/3.639544520803E+01/,    GLN49
     9     GLN(20)/3.933988418720E+01/, GLN(21)/4.233561646075E+01/,    GLN50
     1     GLN(22)/4.538013889848E+01/, GLN(23)/4.847118135184E+01/,    GLN51
     2     GLN(24)/5.160667556776E+01/, GLN(25)/5.478472939811E+01/,    GLN52
     3     GLN(26)/5.800360522298E+01/, GLN(27)/6.126170176100E+01/,    GLN53
     4     GLN(28)/6.455753862701E+01/, GLN(29)/6.788974313718E+01/,    GLN54
     5     GLN(30)/7.125703896717E+01/, GLN(31)/7.465823634883E+01/,    GLN55
     6     GLN(32)/7.809222355332E+01/, GLN(33)/8.155795945612E+01/,    GLN56
     7     GLN(34)/8.505446701758E+01/, GLN(35)/8.858082754220E+01/,    GLN57
     8     GLN(36)/9.213617560369E+01/, GLN(37)/9.571969454214E+01/,    GLN58
     9     GLN(38)/9.933061245479E+01/, GLN(39)/1.029681986145E+02/     GLN59
      DATA GLN(40)/1.066317602606E+02/, GLN(41)/1.103206397148E+02/,    GLN60
     1     GLN(42)/1.140342117815E+02/, GLN(43)/1.177718813997E+02/,    GLN61
     2     GLN(44)/1.215330815154E+02/, GLN(45)/1.253172711494E+02/,    GLN62
     3     GLN(46)/1.291239336391E+02/, GLN(47)/1.329525750356E+02/,    GLN63
     4     GLN(48)/1.368027226373E+02/, GLN(49)/1.406739236482E+02/,    GLN64
     5     GLN(50)/1.445657439463E+02/, GLN(51)/1.484777669518E+02/,    GLN65
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Figure A3. (Sheet 23 of 25)

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     6     GLN(52)/1.524095925845E+02/, GLN(53)/1.563608363031E+02/,    GLN66
     7     GLN(54)/1.603311282166E+02/, GLN(55)/1.643201122632E+02/,    GLN67
     8     GLN(56)/1.683274454484E+02/, GLN(57)/1.723527971392E+02/,    GLN68
     9     GLN(58)/1.763958484070E+02/, GLN(59)/1.804562914175E+02/,    GLN69
     1     GLN(60)/1.845338288614E+02/                                  GLN70
      DATA GLN(61)/1.886281734237E+02/, GLN(62)/1.927390472878E+02/,    GLN71
     1     GLN(63)/1.968661816729E+02/, GLN(64)/2.010093163993E+02/,    GLN72
     2     GLN(65)/2.051681994826E+02/, GLN(66)/2.093425867525E+02/,    GLN73
     3     GLN(67)/2.135322414946E+02/, GLN(68)/2.177369341140E+02/,    GLN74
     4     GLN(69)/2.219564418191E+02/, GLN(70)/2.261905483237E+02/,    GLN75
     5     GLN(71)/2.304390435658E+02/, GLN(72)/2.347017234428E+02/,    GLN76
     6     GLN(73)/2.389783895618E+02/, GLN(74)/2.432688490030E+02/,    GLN77
     7     GLN(75)/2.475729140962E+02/, GLN(76)/2.518904022097E+02/,    GLN78
     8     GLN(77)/2.562211355500E+02/, GLN(78)/2.605649409719E+02/,    GLN79
     9     GLN(79)/2.649216497986E+02/, GLN(80)/2.692910976510E+02/,    GLN80
     1     GLN(81)/2.736731242857E+02/, GLN(82)/2.780675734404E+02/,    GLN81
     2     GLN(83)/2.824742926876E+02/, GLN(84)/2.868931332954E+02/,    GLN82
     3     GLN(85)/2.913239500943E+02/, GLN(86)/2.957666013508E+02/,    GLN83
     4     GLN(87)/3.002209486470E+02/, GLN(88)/3.046868567657E+02/,    GLN84
     5     GLN(89)/3.091641935802E+02/, GLN(90)/3.136528299499E+02/,    GLN85
     6     GLN(91)/3.181526396202E+02/, GLN(92)/3.226634991267E+02/,    GLN86
     7     GLN(93)/3.271852877038E+02/, GLN(94)/3.317178871969E+02/,    GLN87
     8     GLN(95)/3.362611819792E+02/, GLN(96)/3.408150588708E+02/,    GLN88
     9     GLN(97)/3.453794070623E+02/, GLN(98)/3.499541180408E+02/     GLN89
      DATA GLN(99)/3.545390855194E+02/, GLN(100)/3.591342053696E+02/    GLN90
C     ------------------                                               GLN91
    5 NDX=X                                                           GLN92
      T=X-FLOAT(NDX)                                                  GLN93
      IF(T.EQ.0.0) GO TO 51                                           GLN94
      DX=XLIM1-X                                                      GLN95
      IF(DX.LT.0.0) GO TO 40                                          GLN96
C                                                                     GLN97
C     RATIONAL CHEBYSHEV APPROXIMATION ON 2.LT.X.LT.3 FOR GAMMA(X)    GLN98
C                                                                     GLN99
      NXM=NDX-2                                                       GLN100
      PX=PCOE(1)                                                      GLN101
      DO 10 I=2,9                                                     GLN102
   10 PX=T*PX+PCOE(I)                                                 GLN103
      QX=QCOE(1)                                                      GLN104
      DO 15 I=2,4                                                     GLN105
   15 QX=T*QX+QCOE(I)                                                 GLN106
      DGAM=PX/QX                                                      GLN107
      IF(NXM.GT.0) GO TO 22                                           GLN108
      IF(NXM.EQ.0) GO TO 25                                           GLN109
C                                                                     GLN110
C     BACKWARD RECURSION FOR 0.LT.X.LT.2                              GLN111
C                                                                     GLN112
      DGAM=DGAM/(1.+T)                                                GLN113
      IF(NXM.EQ.-1) GO TO 25                                          GLN114
      DGAM=DGAM/T                                                     GLN115
      GAMLN=ALOG(DGAM)                                                GLN116
      RETURN                                                          GLN117
C                                                                     GLN118
C     FORWARD RECURSION FOR 3.LT.X.LT.8                               GLN119
C                                                                     GLN120
```

Figure A3. (Sheet 24 of 25)

```
   22 XX=2.+T                                                          GLN121
      DO 24 I=1,NXM                                                    GLN122
      DGAM=DGAM*XX                                                     GLN123
   24 XX=XX+1.                                                         GLN124
   25 GAMLN=ALOG(DGAM)                                                 GLN125
      RETURN                                                           GLN126
C                                                                      GLN127
C     X.GT.XLIM1                                                       GLN128
C                                                                      GLN129
   40 RX=1./X                                                          GLN130
      RXX=RX*RX                                                        GLN131
      IF((X-XLIM2).LT.0.) GO TO 41                                     GLN132
      PX=Q(1)*RXX+Q(2)                                                 GLN133
      GAMLN=PX*RX+(X-.5)*ALOG(X)-X+RTWPIL                              GLN134
      RETURN                                                           GLN135
C                                                                      GLN136
C     X.LT.XLIM2                                                       GLN137
C                                                                      GLN138
   41 PX=P(1)                                                          GLN139
      SUM=(X-.5)*ALOG(X)-X                                             GLN140
      DO 42 I=2,5                                                      GLN141
      PX=PX*RXX+P(I)                                                   GLN142
   42 CONTINUE                                                         GLN143
      GAMLN=PX*RX+SUM+RTWPIL                                           GLN144
      RETURN                                                           GLN145
C                                                                      GLN146
C     TABLE LOOK UP FOR INTEGER ARGUMENTS LESS THAN OR EQUAL 100.      GLN147
C                                                                      GLN148
   51 IF(NDX.GT.100) GO TO 40                                          GLN149
      GAMLN=GLN(NDX)                                                   GLN150
      RETURN                                                           GLN151
      END                                                              GLN152
```

Figure A3. (Sheet 25 of 25)

## APPENDIX B: NOTATION

| | |
|---|---|
| $a_o$ | Incident wave amplitude |
| $A_o$ | $-iga_o/\omega$ |
| $g$ | Gravitational acceleration |
| $h$ | Water depth |
| $i$ | $\sqrt{-1}$ |
| $J$ | Bessel function of the first kind |
| $k$ | Wave number |
| $n$ | Non-negative integers |
| $r$ | Polar coordinate |
| $t$ | Temporal coordinate |
| $x$ | Horizontal coordinate |
| $y$ | Horizontal coordinate |
| $Y$ | Bessel function of the second kind |
| $z$ | Vertical coordinate |
| $\alpha$ | Incident wave angle |
| $\eta$ | Free surface displacement |
| $\theta$ | Polar coordinate |
| $\theta_o$ | Angle related to wedge angle |
| $\nu$ | Value related to wedge angle |
| $\Phi$ | Velocity potential function |
| $\phi$ | Horizontal component of the velocity potential function |
| $\phi_o$ | Defined in Equation 12 |
| $\phi_i$ | Incident wave velocity potential function |
| $\phi_r$ | Reflected wave velocity potential function |
| $\phi_s$ | Scattered wave velocity potential function |
| $\overline{\phi}$ | Finite cosine transform of $\phi$ |
| $\omega$ | Wave radian frequency |